MATHEMATICS REVIEW 10

EXTREMA OF FUNCTIONS OF TWO VARIABLES

1. We now have a two-dimensional surface rather than a line. Let us assume that our functions are everywhere (partially) differentiable at least twice. At an extremum (peak/valley) the tangent **plane** to the surface will be **horizontal**. Algebraically this means that the first order partials, f_x and f_y, will be simultaneously zero. A **necessary** condition for an extremum is that $f_x = f_y = 0$.

(x_0,y_0) is a **stationary point** of the function and $z_0 = f(x_0,y_0)$ is a stationary/critical value.

2. **Hunting Extrema**: the procedure:

(i) Turn on your brain.

(ii) Write down the function;

e.g., $z = f(x,y) = 2x^2 + 3y^2 + 2x - 2y + 9$. $(x,y) \in R^2$.

(iii) Calculate the first order partial derivatives:

$z_x = 4x + 2$ and $z_y = 6y - 2$.

(iv) Set the partials equal to zero

$z_x = z_y = 0$

or $\quad 4x + 2 = 0$

$6y - 2 = 0.$

(v) Solve the simultaneous equations for x_0, y_0

$x_0 = -1/2 \quad y_0 = 1/3$.

3. Unfortunately our procedures locates **two sorts of extrema** (maxima and minima) and does not provide a way to distinguish between them, it also detects points in the plane where $f_x = f_y = 0$ for which no extremum exists -- **saddle points** (two-dimensional analogues of points of inflection). To work out the possibilities we use the following technique. Define the "*discriminant*" of the function to be $D(x,y) = f_{xx} . f_{yy} - (f_{xy})^2$.

Then

(a) if $D(x_0,y_0) > 0$ and $f_{xx} < 0$ we have a **relative maximum**;

(b) if $D(x_0,y_0) > 0$ and $f_{xx} > 0$ we have a **relative minimum**;

(c) if $D(x_0,y_0) < 0$ we have a **saddle point**;

(d) if $D(x_0,y_0) = 0$ the **procedure fails.**

In our example $z_{xx} = 4$, $z_{xy} = 0$, $z_{yy} = 6$, $z_{yx} = 0 = z_{xy}$, and so

$$D(x_0,y_0) = D(-1/2,1/3) = 4.6 - (0)^2 = 24 > 0$$

which (since $z_{xx} > 0$) means we have a **relative minimum**.

4. Figure 1 provides a schematic representation of critical points, stationary points, and extrema.

5. Assume that

$$f: R^0XR^0 \rightarrow R$$

is a **continuous** function which is **everywhere at least twice continuously differentiable** and which is **strictly concave (convex)**. These assumptions guarantee that the graph of the function is hill (valley) shaped with a unique global maximum (minimum) which lies in the interior of the (x,y) plane (see Figure 2). At an extremum the **tangent plane** to the *two dimensional surface*, G_f, will be horizontal. From an algebraic point of view this means that the first order partial derivatives, f_x and f_y, must be simultaneously zero, i.e. at the stationary point of the function

$$f_x = f_y = 0$$

which is the **first order condition** (necessary condition) for an extremum of a function of two variables. This means that **if** the function achieves a maximum or a minimum at (x_0,y_0) then the equations of the two first order partial derivatives are *simultaneously equal to zero* when we substitute x_0 for x and y_0 for y. Notice that the two first order partial derivatives are **functions** (i.e. they specify rules by which the ordered pairs in the plane can be converted into *unique* values of z – the height of the surface above the relevant point in the plane) but when we set those partial derivatives equal to zero we convert the function rules into a set of two simultaneous **equations** which can be solved for x and y.

6. Unfortunately the first order conditions are only **necessary** for an extremum to exist they are not **sufficient** conditions. The proposition A (e.g. "the real number x is less than two") is a necessary condition for the proposition B (e.g. " the real number x is less than ten") if the condition B can only hold (be a true

statement) if the proposition A holds (is true). This means that if we **know** that B holds then A **must** *necessarily* hold too. On the other hand if A (x is a human being) is a *sufficient* condition for B (x is a male) then **if** A holds B *must* also hold.

The two types of logical conditions have very different implications. In the case of a sufficient condition we *start* from the truth or falsity of **A** since if A is true then B must also be true since A is *by itself* sufficient to guarantee B.

On the other hand with necessary conditions we *start* with **B** since B can *only* occur if A also occurs. Therefore if we know that B is true then A must also be true, however the fact that A occurs does not *guarantee* that B will occur because A is only required for B to be true but it is not *by itself* enough to establish B.

7. Unfortunately $f_i = f_j = 0$ is a **necessary condition** for an extremum **not** a sufficient condition, which means that if there is an extremum at the point in the plane (x_0,y_0) then $f_i = f_j = 0$, but $f_i = f_j = 0$ does not mean that there is an extremum at the point (x_0,y_0). In other words while every extremum satisfies $f_i = f_j = 0$ we can also have **saddle points** (see Figure 3) or a set of **horizontal points of inflection** (see Figure 4). To sort out the possibilities we use the following algebraic technique (which, of course does not require us to plot complicated surfaces and which can be readily generalized to functions of three or even n dimensions).

8. Define the "**discriminant**" of the function f to be $D(x_0,y_0) = f_{xx}f_{yy} - f_{xy}^2$, then if:

(1) $D(x_0,y_0) > 0$ and $f_{xx} < 0$ at (x_0,y_0) we have a **maximum** at (x_0,y_0).

(2) $D(x_0,y_0) > 0$ and $f_{xx} > 0$ at (x_0,y_0) we have a **minimum** at (x_0,y_0).

(3) $D(x_0,y_0) < 0$ at (x_0,y_0) we have a **saddle point** at (x_0,y_0).

(4) $D(x_0,y_0) = 0$ at (x_0,y_0) then the procedure **fails** and we need more information to determine what is happening at (x_0,y_0).

Note that in the case of two or more independent variables the partials must not only exist at the stationary points but they must also be continuous there.

Note that $|H| = \begin{matrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{matrix} = f_{xx}f_{yy} - f_{yx}f_{xy} = f_{xx}f_{yy} - 2f_{xy}$

is the Hessian determinant of the matrix of second order partial derivatives of the function f. We use the Hessian determinants to deal with the higher order cases – of course, if you do not know what a determinant or a matrix is then this will not mean anything to you.

Figure 1

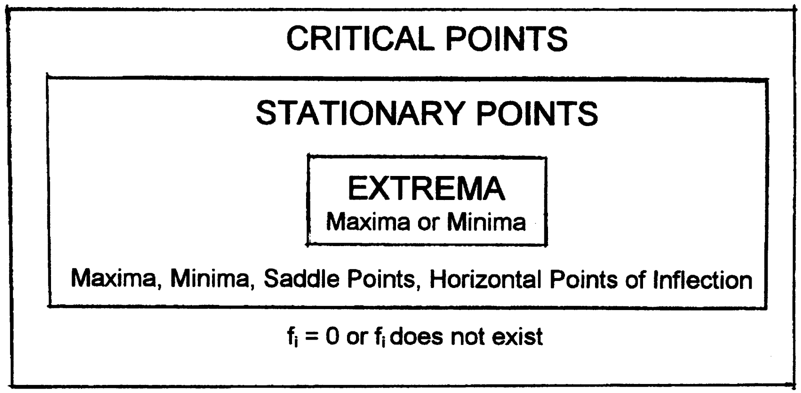
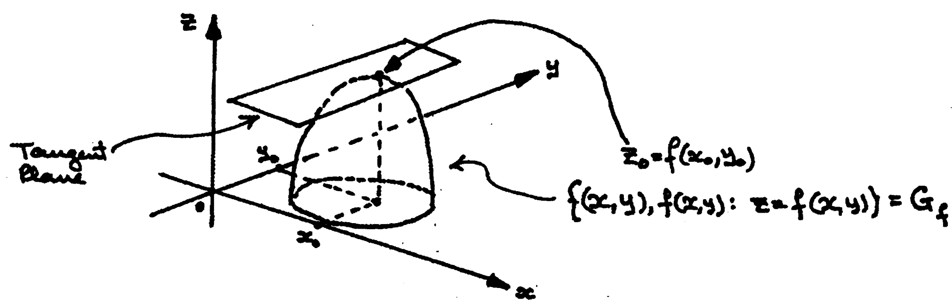


Figure 2



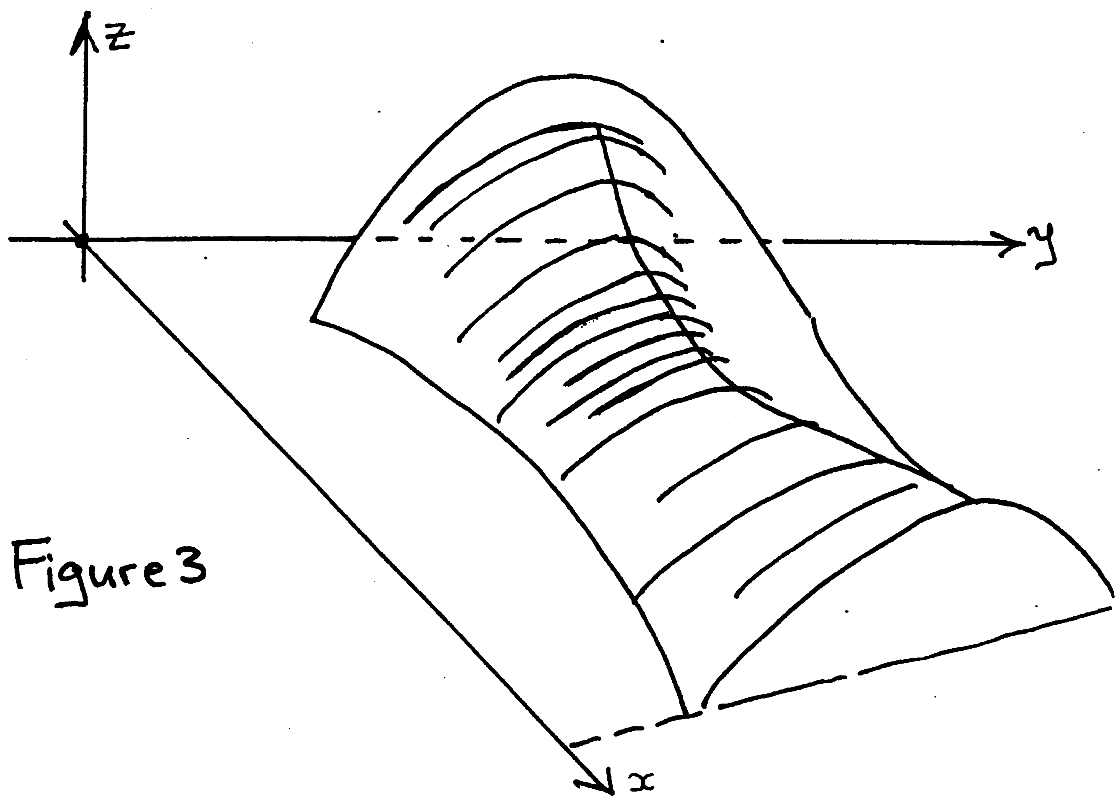


Figure 3

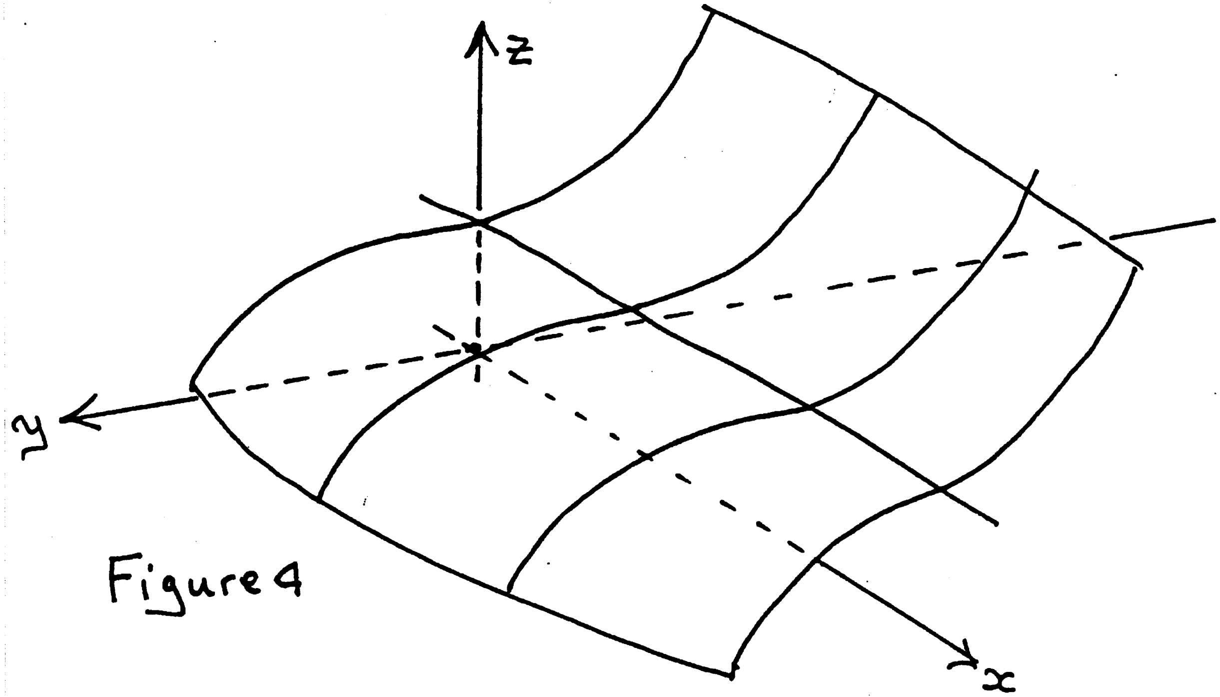


Figure 4

MATHEMATICAL REVIEW 11

FUNCTIONS OF TWO VARIABLES: CONSTRAINED EXTREMIZATION

1. Microeconomics is largely concerned with *scarcity and choice*. Mathematically we represent such problems as **constrained extrema** – maxima or minima that satisfy some constraint: consumers maximize utility subject to budget constraints (associated with finite nominal incomes and positive prices), firms maximize profits subject to demand and technological constraints (embodied in a production function), firms also minimize the costs of production (associated with positive input prices) incurred in the production of the profit maximizing output, universities maximize enrollment subject to financing constraints, etc.

In this general setting we often talk about **objective functions** (the functions to be maximized or minimized) being extremized subject to constraints. The problems take the form

$$\underset{(x_1, \ldots, x_n)}{\text{Max}} \ f(x_1, \ldots, x_n) \qquad \text{or} \qquad \underset{(x_1, \ldots, x_n)}{\text{Min}} \ f(x_1, \ldots, x_n)$$

$$\text{subject to } g(x_1, \ldots, x_n) = c.$$

2. Specific economic examples would be:

$$\underset{(x_1, \ldots, x_n)}{\text{Max}} \ U(x_1, \ldots, x_n) = U$$

$$\text{s.t. } \Sigma P_i x_i = M_0 \quad \text{or} \quad P_1x_1 + P_2x_2 + \ldots + P_nx_n = M_0.$$

or $\quad \text{Max } Q = Q(L,K)$

$$(L,K)$$

$$\text{s.t. } \Sigma\ pj\ xj = w_0L + r_0K = TC_0$$

where we have n = 2 factor inputs in this case.

The firm's profit maximizing choice of inputs – labor and capital – is an example of a two variable constrained extremum. There are two ways in which we can formulate the problem, both of which lead to the same solution. The first approach is to take the production function as the objective function to be maximized subject to the constraint that the firm has a given amount of money (TC_0) to spend and that the labor and capital inputs have given factor prices w_0 and r_0 respectively. The firm will have solved this problem if it achieves a tangency between its given **isocost constraint** and the highest attainable **isoquant** (see Figure 1). At the point of tangency the slope of the isoquant and the slope of the isocost curve are equal. This means that the negative of the ratio of the two marginal products ($-MP_L/MP_K$) is equal to the negative of the ratio of the two factor prices ($-w_0/r_0$). Multiplying both ratios by –1 yields $MP_L/MP_K = w_0/r_0$ or $MP_L/w_0 = MP_K/r_0$. The last of these equations says that the firm will have allocated its resources between labor and capital in an optimal fashion if the last dollar spent on labor gives the same rate of return as the last dollar spent on capital.

The second approach (see Figure 2) to choosing the optimal input mix takes costs as the objective to be **minimized** and the level of output as given. The optimum is now found where the lowest attainable isocost line is tangent to the given isoquant – which turns out to be the highest attainable isoquant in the previous formulation of the problem. The second formulation is called the **dual** of the first formulation of the problem. It should be obvious that the two approaches yield exactly the same solution to the firm's problem of choosing the optimal input mix that will maximize profits.

These are the two diagramatic techniques that I explained in the lecture.

3. One way of solving a constrained extremization problem algebraically is to get rid of the constraint by **substitution**. This will only be possible if we have an *explicit algebraic form* for the constraint and the objective function.

EX:

$$\max_{(x,y)} z = f(x,y) = x^2 + 3xy + y^2$$

$$\text{s.t. } x + y = 100.$$

We can think of the objective function as being a hill (the inverted parabaloid in Figure 3) Obviously if we were unconstrained in our choice of (x,y) commodity bundles we would choose the bundle that would the maximum z, z^*. But our choices are confined to bundles of x and y that lie along (or inside) the linear constraint. This is equivalent to driving a vertical plane through the surface as in Figure 4. Now the highest point that we can attain is z_0^* where $x = x_0^*$ and $y = y_0^*$. In terms of level curves we see in Figure 5 that our unconstrained choice would be z^*, but that we will choose z_0^*, at the tangency between the linear constraint line (with end points, x-bar and y-bar) and the highest attainable level curve z_0^* (at (x_0^*,y_0^*)).

Now we can re-write the constraint as y = 100 - x and reformulate the problem as:

$$\underset{(x,y)}{\text{Max}}\ Z = x^2 + 3x(100-x) + (100-x)^2.$$

You should be able to show that $\frac{dz}{dx} = -2x + 100$ so that

$x_0 = 50$ is a stationary point for the **reformulated problem.** We then solve for $y_0 = 100 - x_0 = 50$ and so our solution to the *original problem* is $(x_0, y_0) = (50, 50)$.

4. *In most cases in economics we cannot use the substitution procedure.* Also in economic theory we are not interested in actually determining the stationary points -- rather we are interested in the information about the nature of the economic agent's equilibrium that is implicit in the solution of the constrained extremization. In these cases we use a mathematical "trick" which was devised by the French mathematician Lagrange (lah granj) and is called the **Lagrange Multiplier** technique. This procedure also converts the original intractable problem (extremization subject to a constraint) into the easier problem of extremizing an *unconstrained* function.

EX. Say we wish to apply this technique to our problem already solved via substitution. Then we proceed as follows:

(a) First set up the so-called Lagrangian Function

$$£(x,y,\lambda) = f(x,y) - \lambda[x + y - 100]$$

Notice that the Lagrangian, £, is a function of the *three* variables: x, y, and the Lagrange multiplier, λ. Also note that the Lagrangian is the sum of the original function (f(x,y)) and the expression $\lambda[x + y - 100]$ which is the **product** of the unknown, λ, and the term in square parentheses. Observe that when the constraint is satisfied the term in square brackets will be zero. Also note that it doesn't matter whether we add or subtract the second term.

(b) We now have an unconstrained maximization problem to solve; vis.

$$\max_{(x,y,\lambda)} \quad £(x,y,\lambda) = x^2 - xy + y^2 - \lambda[x + y - 100].$$

We proceed by calculating the *three* first order partials:

$$£_X = 2x - y - \lambda$$

$$£_Y = -x + 2y - \lambda$$

$$£_\lambda = 100 - x - y.$$

(c) Set these partials (simultaneously) equal to zero and solve the three equations for the three unknowns: x_0, y_0, λ_0; i.e.,

$$£x = £y = £_\lambda = 0$$

$$\Rightarrow \quad \begin{aligned} £x &= 2x - y - \lambda = 0 \\ £y &= -x + 2y - \lambda = 0 \\ £_\lambda &= 100 - x - y = 0 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} 2x - y &= \lambda \\ -x + 2y &= \lambda \\ x + y &= 100 \end{aligned}$$

Notice that **if** $£(x,y,\lambda)$ is extremized **then** the constraint $(x + y = 100)$ *will* be satisfied. The solution of our Lagrangian problem is $x_0 = y_0 = \lambda_0 = 50$.

(d) We should now check the second-order conditions (which are laid out in section 5 below) to make sure we have a maximum. However, in an economics context we can **assume** that the economic agent is in equilibrium and presumably can tell the difference between a maximum and a minimum! Finally we should evaluate the function at the stationary point; i.e., $z_0 = f(x_0,y_0) = 2500$.

4. We will know do a constrained maximization problem taken from economics. A household is in equilibrium when it is purchasing that bundle of goods which satisfies its budget constraint and maximizes the household's utility. At that equilibrium the household will be consuming a commodity bundle which is on the highest attainable indifference curve. At that point in the commodity space there will be tangency between the indifference curve and the budget constraint.

The household's problem may be formulated as:

$$\max_{(x,y)} \quad U = U(x,y) \quad (x,y) \in (R^0)^2 \quad U(0,0) = 0,\ U_X, U_Y > 0,\ U_{XX}, U_{YY} < 0$$

s.t. $P_x x + P_y y = M_0$.

The appropriate Lagrangian function is:

$$£(x,y,\lambda) = U(x,y) - \lambda(M_0 - P_x x - P_y y).$$

The first order conditions are:

$$\frac{\partial £}{\partial x} = £_X = \frac{\partial U}{\partial x} - \lambda P_X = U_X - \lambda P_X = 0$$

$$\frac{\partial £}{\partial y} = £_y = \frac{\partial U}{\partial y} - \lambda P_y = U_y - \lambda P_y = 0$$

$$\frac{\partial £}{\partial \lambda} = £_\lambda = \qquad M_0 - P_x x - P_Y y = 0$$

From which we obtain:

$$U_X = MU_X = \lambda P_X$$

$$U_y = MU_y = \lambda P_y$$

$$P_x x + P_y y = M_0$$

and so the budget constraint is satisfied at the stationary point. Solving the first two equations yields

$$\frac{U_X}{U_Y} = \frac{P_x}{P_y} \quad \text{or} \quad \frac{U_X}{P_X} = \frac{U_Y}{P_Y} = \lambda$$

where $\lambda = \frac{\partial U}{\partial M_0}$ the rate of change of utility associated with a

relaxation of the budget constraint.

Note that we are not able to determining x_0, y_0 explicitly without more information.

Second Order Conditions for a Constrained Extrema.

(1) $f_{xx}(g_y)^2 - 2f_{xy}g_xg_y + f_{yy}(g_x)^2 < 0$ then f is **maximized.**
(2) $f_{xx}(g_y)^2 - 2f_{xy}g_xg_y + f_{yy}(g_x)^2 > 0$ then f is **minimized.**
(3) $f_{xx}(g_y)^2 - 2f_{xy}g_xg_y + f_{yy}(g_x)^2 = 0$ then the test fails.

Where g_x etc. is the partial derivative of the constraint with respect to x, etc. This second order condition can be written as a bordered Hessian – which we will leave to ECON 406 where you will do constrained extremization in some depth.