MATHEMATICS REVIEW 9

FUNCTIONS OF TWO VARIABLES

1. Functions of a Single Variable (Refer to the LHS of Figure 1.)

$$f: R \to R$$

$$\text{s.t. } y = f(x).$$

In this case the **domain** of f (D_f) and the **codomain** of f (C_f) are both sets of real **numbers**. This function *maps* the real line to the real line. The *function machine* converts real x's into *unique* real y's.

$$x \to \mid f \mid \to y = f(x) = a + bx + cx^2.$$

(The f is supposed to be in a box.)

The *graph* of f (G_f) is a (one-dimensional) **line** in a two-dimensional space -- the so-called Cartesian plane, $RXR = R^2$. Each variable must have its own axis -- a real line -- with the axes (usually) drawn at right angles to one another and (usually) with the same scale. Where $f(x_0)$ is the *height* of the graph above the x axis at $x = x_0$.

2. **Functions of Two Variables** (Refer to the RHS of Figure 1.)

$$f: R^2 \to R$$

$$\text{s.t. } z = f(x,y)$$

In this case the **domain** of f (Df) is a set of *ordered pairs* of real numbers (x,y)s which correspond to points in the real Cartesian plane ($R \times R = R^2$) and the **codomain** of f (Cf) corresponds to the real line. This function *maps* the real **plane** to the real **line**. The function machine converts **pairs** of x's and y's into (unique) real **numbers** (the z's).

$$(x,y) \rightarrow \mid f \mid \rightarrow z = f(x,y) = 2 + 3x - 4xy + 3y^2$$

$$(x,y) \rightarrow \mid f \mid \rightarrow z = f(x,y) = a + bx + cxy - ky^2$$

The graph of f is a two-dimensional **surface** in a three-dimensional space (one dimension for each variable). Where $f(x_0,y_0)$ is the height of the surface above the (x,y) plane at the point with coordinates $x = x_0$, and $y = y_0$.

A useful way to think about functions of two variables is to think of the (x,y) plane as being covered by a **grid**, where each intersection corresponds to a particular pair of real numbers. We can then think of a number being attached to each intersection point so that, for example, if we are plotting the first function above, the grid point (2,3) has the number 11 attached to it and the grid point (3,2) would have the number -1 attached to it. These numbers represent the *height* of the surface above the point in question. (See the third perspective) diagram in Figure 1 and the table at the top of page 133 of the handout.)

We can also represent the surface by drawing its **contours** (which mathematicians call **level curves**); i.e., we can join up all points in the plane (*all (x,y) pairs*) which have the same *z* coordinate. We can *slice* through the surface *parallel* to the (x,y) plane (we can "decapitate" the surface) and *project* the "rim" of the *cut* down onto the plane to see which (x,y) points in the **plane** correspond to the points on the **surface** at a *given height*. (Compare with height above sea-level **contours**, isotherms and isobars on meteorological maps). $L = \{(x,y) : f(x,y) = z_0\}$ is the set of points in the **(x,y) plane** which lie **beneath** points on the *surface* with the *given height (z_0).* L is the level curve in the plane that corresponds to the function taking the value z_0. (Figures 2a – 2c go about here). See also Figure 6 p.134 which shows lines of constant height on the surface of the production function, Q=100 and Q=120, and Figure 7 on the bottom half of p.135.)

3. **Functions of N Variables**

From an algebraic point of view these present no new problems, but they cannot be handled easily, or at all, with conventional perspective diagrams. A function of n variables $(x_1, x_2, \ldots, x_n)$ is a mapping from the set of (real) **n-tuples** (*points in an n-dimensional space*) to the **real line** where each image is unique:

$$f: (x_1, x_2, \ldots, x_n) \rightarrow y = f(x_1, x_2, \ldots, x_n) \quad ((x_1, \ldots, x_n) \in R^n)$$

In many cases the rule for finding the image (given the element in the domain) will be an algebraic equation. Note that almost all economic problems are described in terms of functions of n variables (e.g., the consumer's choice of an optimal commodity bundle consisting of n goods and services, the firm's choice of the optimal amount of each of n inputs to employ).

The **graph** of a function of n variables is an n-dimensional *hyper-surface* in a n+1 dimensional space. We can represent an n-dimensional surface in two dimensions by taking a **cross-section** through the surface and projecting the "shadow" (what mathematicians call a **trace**) of the cross-section onto the two-dimensional plane we are interested in, **holding all other variables constant**. That is, we cut through the surface *parallel* *to one of the axes* (we insert a plane into the space parallel to one of the axes) and then *project the image of the cut* onto the plane. This yields a conventional looking graph (line).

Returning to our "grid" interpretation of the function of two variables, What we mean by a *cross-section* through the surface is a *cut* through the surface *along* one of the *grid lines* i.e. along a line at which **one of the variable is being held constant.** For example, we could cut along the line parallel to the y-axis with x held constant at 2, or at the level x=3, or we could cut along a line parallel to the x axis holding y constant at y=4 or y=9.7. (The diagram on p. 135 shows cuts through the corn production function parallel to the Nitrogen axis, holding Phosphate constant at the levels P=40 and P=80. The top part of the diagram on p. 135 shows the graphs of these cross-sections. The top part of the graph on p. 136 shows a typical short-run Total Product of Labor curve that is a cross-section through the production surface holding capital constant.)

Up until this point we have been treating our demand and supply functions as functions of a single variable, price, but we know that these functions really have many independent variables. For example we know that the demand function is a function of the form:

$Q^d{}_X = f(P_X, P_S, P_C, Y)$ $((P_X, P_S, P_C, Y) \in R^n$ P_X etc. ≥ 0)

where P_x is the price of x, P_s is the price of a substitute, P_c is the price w of a complement, and Y is real income. The conventional **demand curve** is the *shadow* (trace) *of the graph of the* **demand function** where *all of the other variables* (other than $Q^d{}_X$ and P_X) *are held constant,* i.e. we **fix** the values of the other variables to obtain a specific cross-section. (The graph of the demand function needs one axis for each of the variables -- including the dependent variable. We therefore expect the demand surface to be an n-dimensional surface.) If any one or more of the independent variables changes its value then we move to a new cross-section and the demand curve *seems* to shift.

4. **Partial Derivatives**. A **partial derivative** is a measure of the **slope** of the cross-section through the graph of the function; it measures the **rate of change** of the *dependent* variable with respect to **one** of the *independent variables,* holding *all* of the other independent variables **constant. Partial derivatives** are **marginals** just like ordinary derivatives.

If $\quad z = f(x,y) \quad ((x,y) \in R^2)$

then $\quad \dfrac{\partial z}{\partial x} = f_x(x,y) = \lim\limits_{h \to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$ **if** it exists

and $\quad \dfrac{\partial z}{\partial y} = f_y(x,y) = \lim\limits_{j \to 0} \dfrac{f(x,y+j) - f(x,y)}{j}$ **if** it exists

Let $\quad z = 2 + 3x - 4y$

then $\dfrac{\partial z}{\partial x} = 3 = f_x(x,y)$ and $\dfrac{\partial z}{\partial y} = -4 = f_y(x,y)$.

Let $z = f(x,y) = a + bx + cy + exy + mx^2y + nxy^2 + rx^2y^3$

then $z_x = f_x(x,y) = b + ey + 2mxy + ny^2 + 2rxy3 = \dfrac{\partial z}{\partial x}$

and $z_y = f_y(x,y) = c + ex + mx^2 + 2nxy + 3rx^2y^2 = \dfrac{\partial z}{\partial y}$.

Note that while partial differentiation looks complicated it is actually a straightforward extension of single variable differential calculus, using the standard rules (so there is nothing new for you to learn!) and *treating all variables*, **except the one with which you are differentiating**, as *additive* or *multiplicative constants*. These constants are easily handled during differentiation (they drop out if they are *additive* constants and stay in unchanged if they are *multiplicative* constants). The only problem with partial differentiation is that there is a lot more work to do and you have to keep track of which things are varying and which are constant.

5. **Higher order partial derivatives** are also easily computed. Note that if z is a function of *two* variables (x and y), so that z = f(x,y), then f has **two** *first order* partials (f_x, f_y) and **four** *second order* partials -- f_{xx} and f_{yy} and the two **cross-partials**, f_{xy} and f_{yx}. f_{xy} and f_{yx} are usually equal; i.e., $f_{xy} = f_{yx}$ for all of the functions you are likely to meet with in economics contexts. The second order partials have straightforward interpretations -- they represent the **curvature** of the cross-section graph, just as the first-order partials represent the slope of the cross-section graph. That is, second order partial derivatives tell us how the **first order derivative** *changes* as we ***increase*** the independent variable - *holding all of the other independent variables constant*. The second order derivative tells us about the curvature (concavity or convexity) of the **trace -- the graph of the cross-section**. The second order partials tell us the **slope of the first order partials -** and therefore the **slope** of the **marginal functions.**

$$\frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \qquad \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Since f is a function of *both* x and y we can also calculate second order cross-partial derivatives that tell us, say, how the *derivative* in the **x** *direction* changes as we increase the **y** independent *variable*. That is, this cross partial derivative tells us how the *slope* in the **x direction** changes as we ***shift the cross-section*** so that the new "fixed" value of **y** is *larger* than before (and vice versa for the other cross partial derivative).

$$\frac{\partial^2 z}{\partial x \partial y} = f_{yx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \qquad \frac{\partial^2 z}{\partial y \partial x} = f_{xy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

For the type of "well-behaved" function that we usually encounter in economics the cross partial derivatives will be equal.

Let $\quad z = f(x,y) = a + bx^2y + cxy^2 \quad (x,y) \in R^2$

then $\frac{\partial z}{\partial x} = f_x(x,y) = 2bxy + cy^2 \qquad \frac{\partial z}{\partial y} = f_y(x,y) = bx^2 + 2cxy$

and $f_{xx}(x,y) = \frac{\partial^2 z}{\partial x^2} = 2by \qquad f_{yy}(x,y) = \frac{\partial^2 z}{\partial y^2} = 2cx$

and $f_{xy}(x,y) = \frac{\partial^2 z}{\partial y \partial x} = 2bx + 2cy = f_{yx}(x,y) = \frac{\partial^2 z}{\partial x \partial y}$.

Note that the cross partials *are* equal and so we have either got the answer right or we have made canceling algebraic errors!

6. **The Cobb-Douglas Production Function.**

If $Q = f(L,K) = AL^aK^b$, $A > 0$, $0 < a, \; b < 1$, $f(0,0)=0 \quad ((L,K) \in (R^+)^2$

then $\frac{\partial Q}{\partial L} = aAL^{a-1}K^b = aAL^aL^{-1}K^b = aAL^aK^bL^{-1} = \frac{a(AL^aK^b)}{L}$

$$\frac{\partial Q}{\partial L} = a\left(\frac{Q}{L}\right) = aAP_L = MP_L > 0,$$

and $$\frac{\partial Q}{\partial K} = MP_K = bAL^aK^{b-1} = b(AL^aK^b)/K = b\left(\frac{Q}{K}\right) = bAP_K > 0.$$

The *second order partials* measure the **slopes** of the **marginal** functions:

$$\frac{\partial^2 Q}{\partial L^2} = \frac{\partial^2 aAL^{a-1}K^b}{\partial L^2} = (a-1)aAL^{a-2}K^b = a(a-1)AL^aK^bL^{-2}$$

$$= \frac{a(a-1)(AL^aK^b)}{L^2} = a(a-1)Q/L^2 < 0.$$

$$\frac{\partial^2 Q}{\partial K^2} = \frac{\partial bA\,L^aK^{b-1}}{\partial K^2} = (b-1)bAL^a\,K^{b-2} = b(b-1)AL^aK^bK^{-2}$$

$$= \frac{b(b-1)AL^aK^b}{K^2} = b(b-1)Q/K^2 < 0.$$

Further

$$\frac{\partial^3 Q}{\partial L^3} = \frac{\partial a(a-1)AL^{a-2}K^b}{\partial L} = (a-2)a\,(a-1)AL^{a-3}K^b$$

$$= a(a-1)(a-2)AL^aK^bL^{-3}$$

$$= \frac{a(a-1)(a-2)(AL^aK^b)}{L^3} = [a(a-1)(a-2)Q]/L^3 < 0.$$

$$\frac{\partial^3 Q}{\partial K^3} = \frac{\partial b(b-1)A\,L^aK^{b-2}}{\partial K} = (b-2)b(b-1)AL^a\,K^{b-3}$$

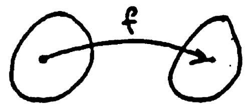
$$= b(b-1)(b-2)AL^aK^bK^{-3}$$

$$= \frac{b(b-1)(b-2)(AL^aK^b)}{K^3} = [b(b-1)(b-2)Q]/K^3 < 0,$$

and so the marginal product curves of the Cobb-Douglas production function are proportional to their average product curves (with the factors of proportionality being the respective exponents), are strictly positive, and have negative and algebraically increasing slopes.

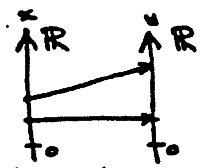
FUNCTION of a SINGLE VARIABLE

$y = f(x) \quad (x \in \mathbb{R})$



$\mathcal{D}_f$ $\mathcal{C}_f$

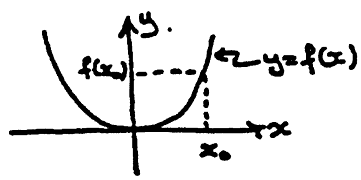
(numbers) (numbers)



Mapping from the real line to the real line

$x \longrightarrow \boxed{f} \longrightarrow y = a + bx + cx^2$

Function machine converts real x_0 into unique real y_0.

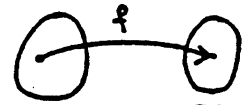


Graph is a (one-dimensional) line in a (two-dimensional) space – the Cartesian plane. $f(x_0)$ is the height of the line above the x axis at $x = x_0$

Figure 1

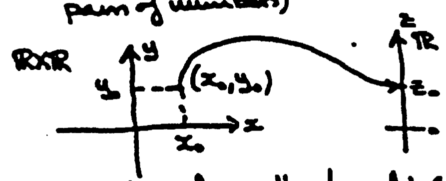
FUNCTION OF TWO VARIABLES

$z = f(x,y) \quad ((x,y) \in \mathbb{R}^2)$



$\mathcal{D}_f$ $\mathcal{C}_f$

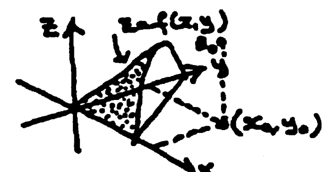
(ordered pair of numbers) (numbers)



Mapping from the (real) Cartesian plane to the real line.

$(x,y) \longrightarrow \boxed{f} \longrightarrow z = a + bx + cy + dx^2 + ey^2 + gxy$

Function machine converts pairs of x_0 and y_0 into real z_0.



Perspective diagram

Graph is a (two-dimensional) surface in a (three-dimensional) space (one dimension for each variable). $f(x_0, y_0)$ is the height of the surface above the (x,y) plane at the point with coordinates $x = x_0$, $y = y_0$.

We can also represent the surface by drawing contours (level surfaces) i.e. we can join up all points in the plane (all

Level curves often have interesting physical interpretations. For example, surveyors draw *topographic maps* that use level curves to represent points having equal altitude. Here $f(x, y)$ = the altitude at point (x, y). Figure 2a shows the graph of $f(x, y)$ for a typical hilly region. Figure 2b shows the level curves corresponding to various altitudes.

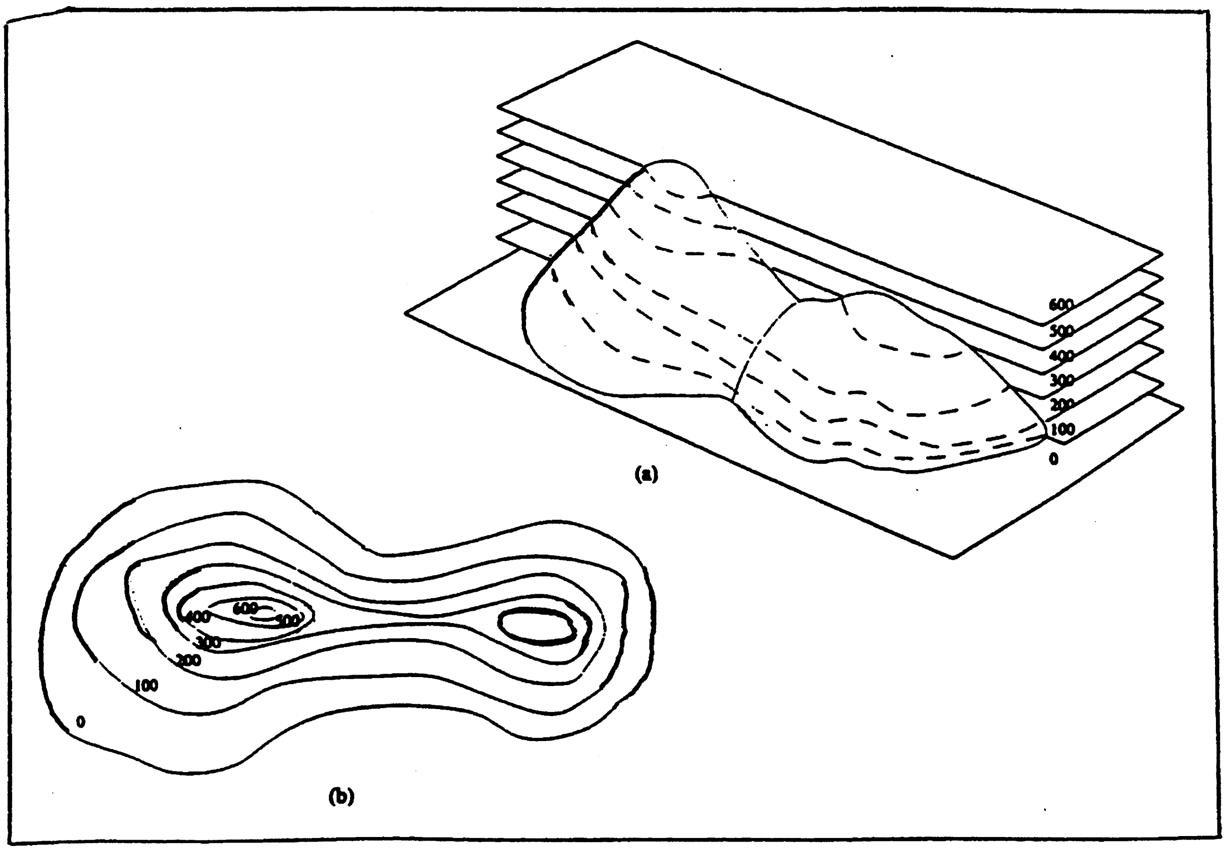
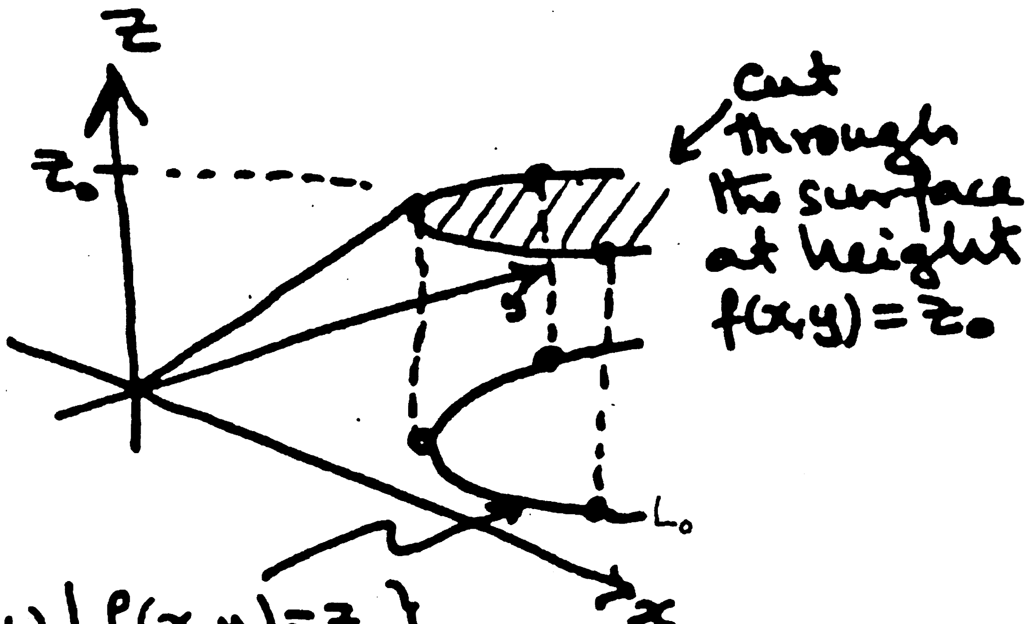


Figure 2(a)&(b)



$L_0 = \{(x,y) \mid f(x,y) = z_0\}$

the set of points in the (x,y) plane which lie beneath points on the surface with the given height (z_0).

Figure 2c

Graph of a General, Hill-shaped Function

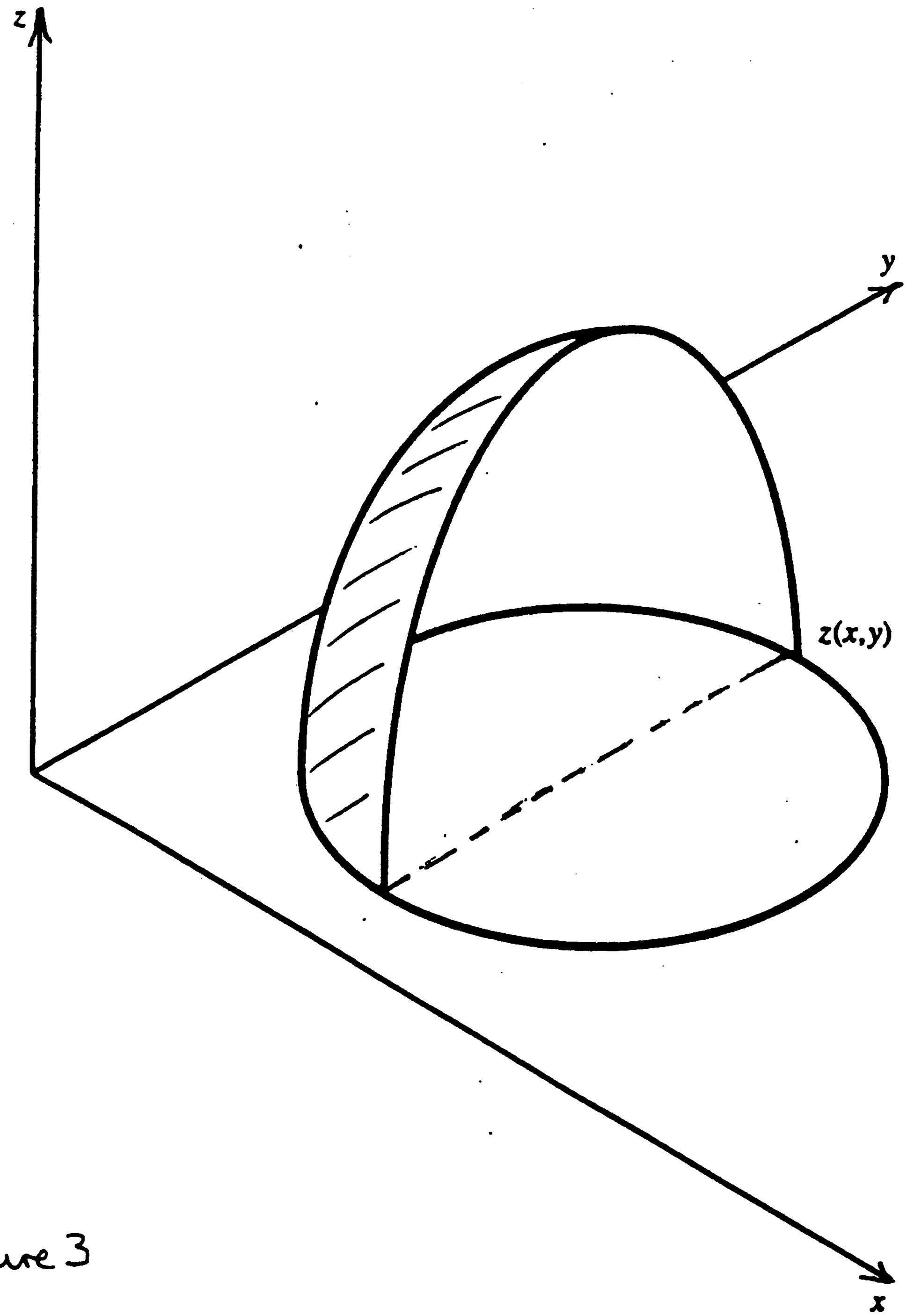


Figure 3

Illustration of the Partial Derivative of z With Respect To x

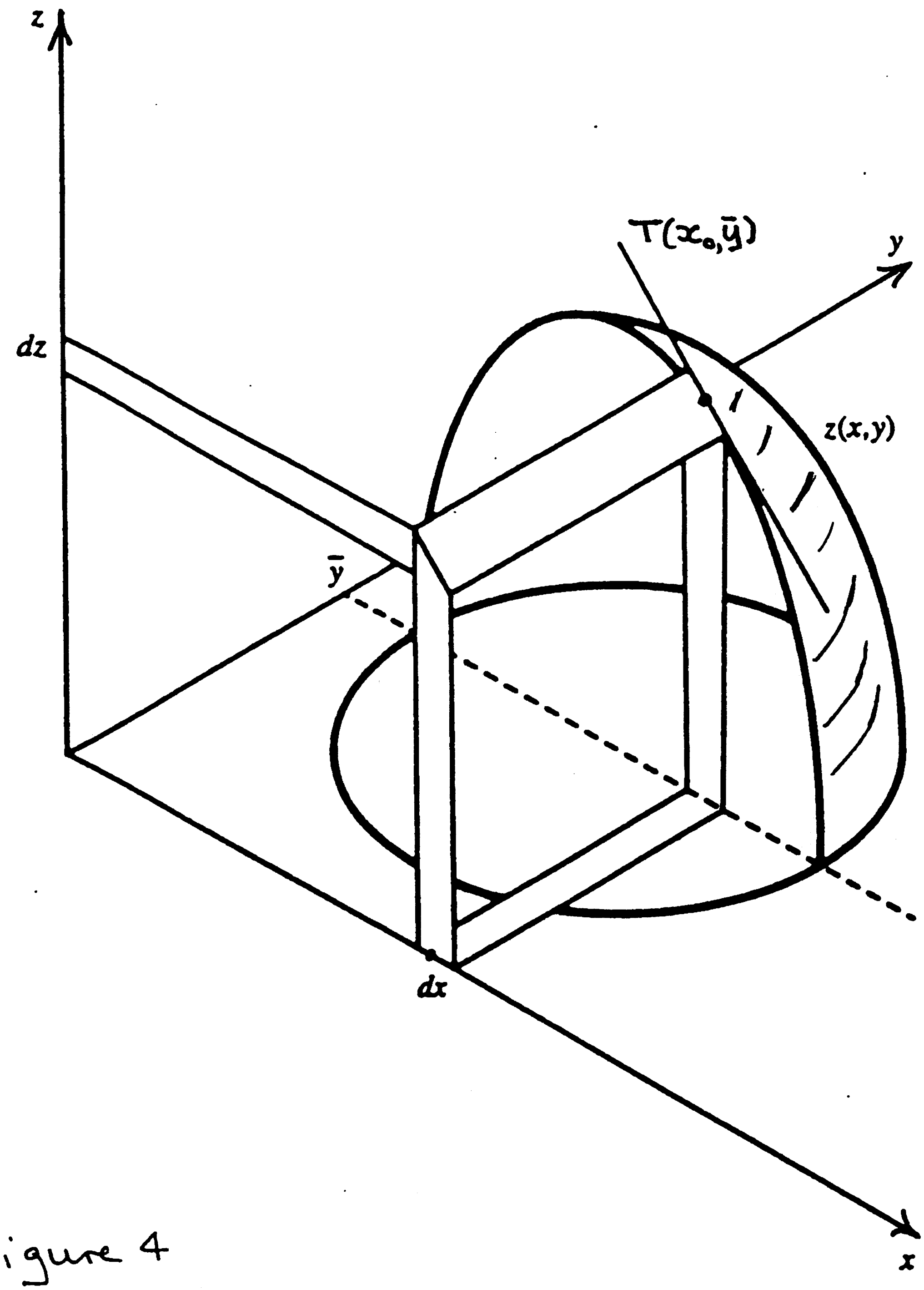


Figure 4

Illustration of the Partial Derivative of z With Respect to y

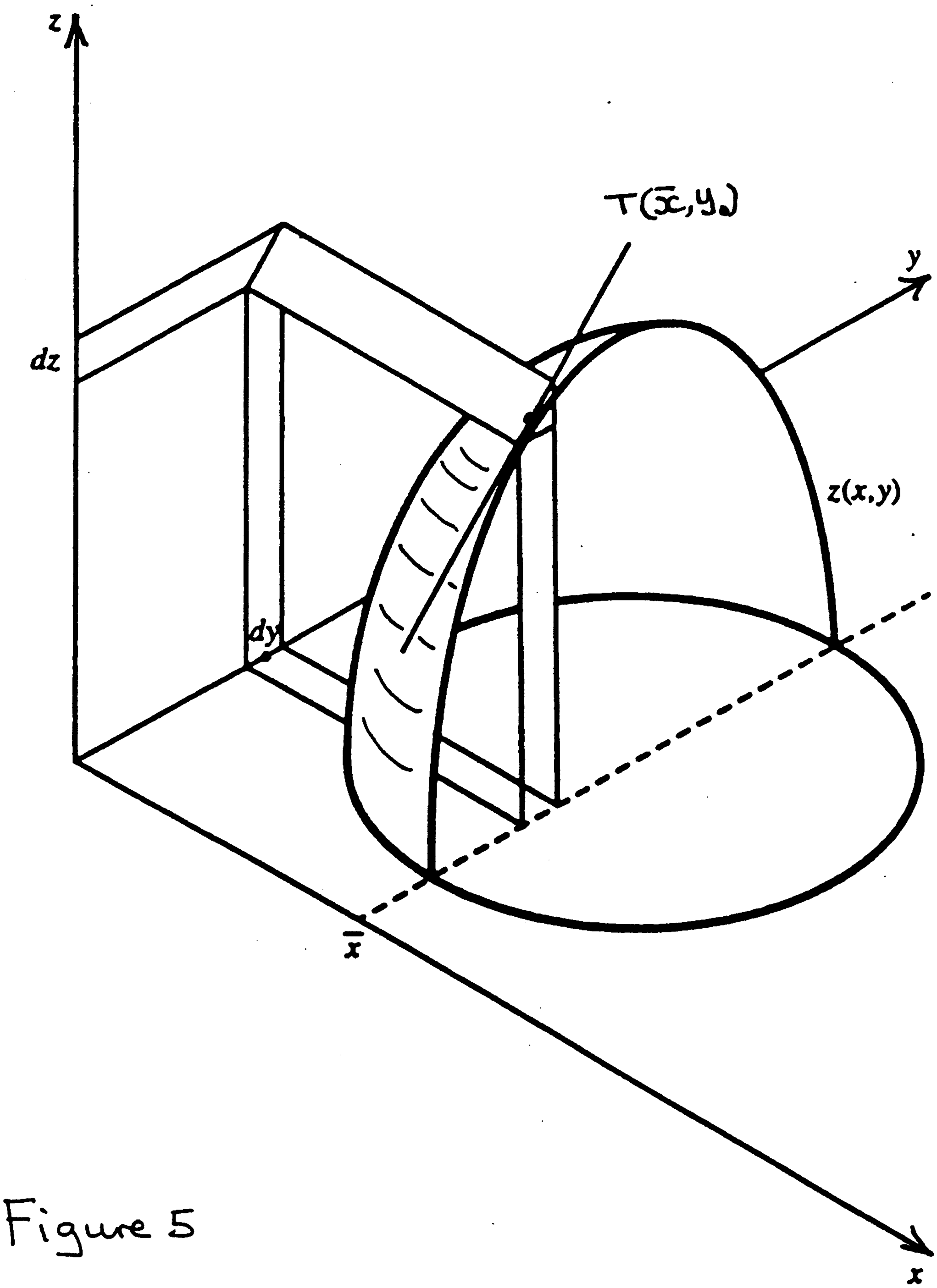


Figure 5