MATHEMATICAL REVIEW 8

EXTREMA OF FUNCTIONS OF ONE VARIABLE

1. **Terminology**. Extrema are maxima or minima (-"a" plural; "um" singular). Values of x at which an extremum or horizontal point of inflection occur are called **critical values** of the function.

(i) A point $(x_0,f(x_0))$ on the graph of f is a **global** maximum of f **if** $f(x_0) \geq f(x)$ for **all** $x \in D_f$. A point is a **local** maximum of f **if** $f(x_0) \geq f(x)$ for all x in the "neighborhood" of x_0 (i.e. some "small" open interval including x_0).

(ii) x_0 is a **stationary value** of f if $\mathbf{f'(x_0) = 0}$, i.e. the graph of the function is perfectly flat at the **stationary point** $(x_0,f(x_0))$. x_0 is called an **extreme value** of f if f has an extremum at x_0 but $f'(x) \neq 0$, say at a "sharp point" (cusp) of the graph of f.

[Figure 1 goes here.]

2. From the diagram we see that extrema occur at stationary points and at points where f'(x) does not exist (f is not differentiable at "sharp" points and at the end points of the interval). We distinguish between **interior** extrema and **end-point** extrema. Economists usually restrict the problem at hand so that all extrema of interest are interior extrema.

3. Even if all extrema are interior and the function is everywhere differentiable we still have to decide whether we have a maximum or minimum. In an economic context the assumption that the decision-maker is in **equilibrium** is usually sufficient to handle

this dilemma. Economists also usually *impose* restrictions on the functions being extremized in order to ensure that any extremum is also a *unique global* extremum, e.g. by *assuming* that the function f is *concave from below* in the case of *maximization* problems. The economic *theorist* can play God within her model and can make (economically sensible) assumptions that generate the sort of solutions that we are looking for – sometimes not knowing very much about the specifics of the situation being modeled brings its own rewards! Of course, ultimately the model has to be confronted with systematic testing, and it is at the testing stage of the analysis that we determine how reasonable our assumptions really are.

There is a further problem -- we may not have an extremum at a stationary point, we may simply have found an **horizontal point of inflection** (i.e., a point where the function alters its curvature from, say, positive to negative).

4. **The Second Derivative Test**: *Assume* that f is everywhere at least twice differentiable and that all extrema are interior extrema.

(i) Locate all stationary points by setting $f'(x) = 0$ and solving the resulting equation for x. The roots of the equation are the stationary points and the corresponding function images ($f(x_0)$) are potential extrema.

(ii) Differentiate a second time and evaluate $f''(x)$ at *each* of the stationary points, x_0. If $\mathbf{f''(x_0) > 0}$ we have a **minimum**, if $\mathbf{f''(x_0) < 0}$ we have a **maximum**, while if $\mathbf{f''(x) = 0}$ the test is **indeterminate** (this does **not** mean we have **necessarily** obtained a horizontal point of inflection).

5. **The Nth. Derivative Test**: *Assume* that f is everywhere differentiable and that we only have interior extrema.

(i) Locate the stationary points by setting $f'(x) = 0$, etc.

(ii) Differentiate a second, third, ..., nth. time (stopping as soon as you obtain a non-zero derivative -- **each of the stationary points has to be evaluated separately**).

(a) If the *order* of the first *non*-zero derivative is **even** then we have an **extremum**. If the derivative, when evaluated at the stationary point, is **negative** then the extremum is a **maximum**; if the derivative, when evaluated at the stationary point, is **positive** then the extremum is a **minimum**.

(b) If the first non-zero derivative is of **odd** order then we have located an horizontal point of inflection.

[Figure 2 goes here.]

6. Let $y = f(x) = x^4$ $(x \in R)$ then we know that this function has a minimum at $x = 0$ (it's the square of a quadratic and so its graph is a "squished" parabola).

$$y' = 4x^3 \quad y' = 0 \Rightarrow 4x^3 = 0 \Rightarrow x_0 = 0$$

and so x_0 is the only stationary point. $y'' = 12x^2$ therefore $y''(0) = 0$; $y''' = 24x$ therefore $y'''(0) = 0$; $y^{IV} = 24$ therefore $y^{IV}(0) = 24 \neq 0$ (i.e. we have at last found a derivative that , evaluated at the stationary point, gives a non-zero value: $y^{IV}(0) = 24 > 0$). This first non-zero derivative (evaluated at x_0) was of the fourth order, and since four is an **even** number there is an **extremum** at $x = x_0 = 0$. Since $f^{IV}(x_0) > 0$ (**positive**) we have a **minimum**.

7. If, like mathematicians, physicist, or engineers, we are "hunting extrema" all we have to do is to evaluate f, which will posses a *known algebraic form*, at all x which are stationary points or where f'(x) fails to exist, and at the two end points of the interval.

8. In practice economists almost never know the precise algebraic form of the functions they are considering. The economist therefore *assumes* that the economic agent is already at an extremum (i.e., the agent has, perhaps by trial and error, discovered the maximum or minimum). This is what an *equilibrium* situation usually means in economics. The economist then uses the **first** and **second order conditions** to deduce the properties of the equilibrium and how it will change if we change any of the parameters of the model (*QCS*). The **first order condition** says that the derivative of the function must be zero at an extremum; the **second order condition** says that the function must be concave down for a maximum and convex down for a minimum ($f'(x)<0$, $f''(x)>0$ respectively). So the economist approaches the extremum issue from the opposite pole to the physicist or engineer. We do not expect to be able to actually find extrema, but instead use the assumptions of extremization, stable preferences and technologies, and equilibrium to generate information about what we may expect to observe in the real world. Only if those predictions fail systematic tests – not just casual empiricism – will the economist abandon her model.

9. Let the firm be ***profit maximizing*** **and** *in equilibrium*. Then $\Pi'(Q) = R'(Q)-C'(Q) = 0$, **i.e. MR = MC** when the firm is in equilibrium -- which is why we locate the firm's profit maximizing output at Q_0 where $MR(Q_0)= MC(Q_0)$. We *assume* that the firm is able to distinguish between a maximum and a minimum, say by a

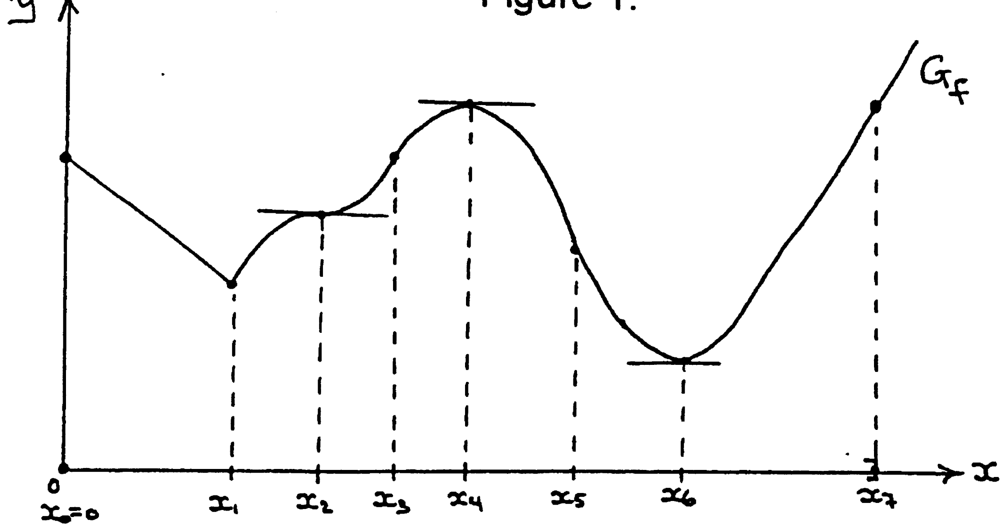
process of trial and error, and so we have $\Pi''(Q_0) < 0$ and hence $R''(Q_0) < C''(Q_0)$ this means that the slope of the MR curve must be less than the slope of the MC curve, and the MC curve must cut the MR curve from below at Q_0.

[Figure 3 goes here.]

10. Let AC = C(Q)/Q ($Q \in R^+$) then AC reaches its minimum where dAC/dQ = (MC-AC)/Q = 0, i.e. where MC=AC since $Q \neq 0$. Further d^2AC/dQ^2 must be **positive** since AC is **minimized.** But d^2AC/dQ^2 = {d(MC-AC)/dQ.Q - (MC-AC).1}/Q^2 = (dMC/dQ - dAC/dQ) / Q because, at the equilibrium MC=AC, and so the negative term in the numerator drops out. Since d^2AC/dQ^2 = (dMC/dQ - dAC/dQ) / Q > 0 the MC curve must cut the AC curve from below at the minimum point of the AC curve.

[Figure 4 goes here.]

Figure 1.



$x \in R^0$ and $x \in [0,x_7]$ (this is a "closed interval" i.e. containing its end points $x_0 = 0$ and $x_7 > 0$, so that f(x) is defined for $x_0 \leq x \leq x_7$). **Extreme points** occur at x_0, x_1, x_4, x_6, and x_7, but x_0 and x_7 are end point extrema where f'(x) fails to exist. Similarly f'(x) does not exist at x_1 (where the graph of f exhibits a sharp point or cusp). We will not concern ourselves with extrema of either of these types in 208 where we will **assume** that our functions are continuous and everywhere at least twice differentiable – and when dealing with marginal functions we will assume that the original total function is everywhere thrice differentiable. **Stationary points** occur at x_2, x_4, and x_6 where f'(x) = 0. x_4 and x_6 are extrema (a maximum and a minimum respectively). x_2 is an horizontal point of inflection. Non – horizontal points of inflection occur at x_3 and x_5. Local extrema occur at x_0, x_1, x_4, x_6, and x_7. The global maximum occurs at x_7, and the global minimum at x_6. x_4 is the global maximum amongst the interior optima.

Figure 2.

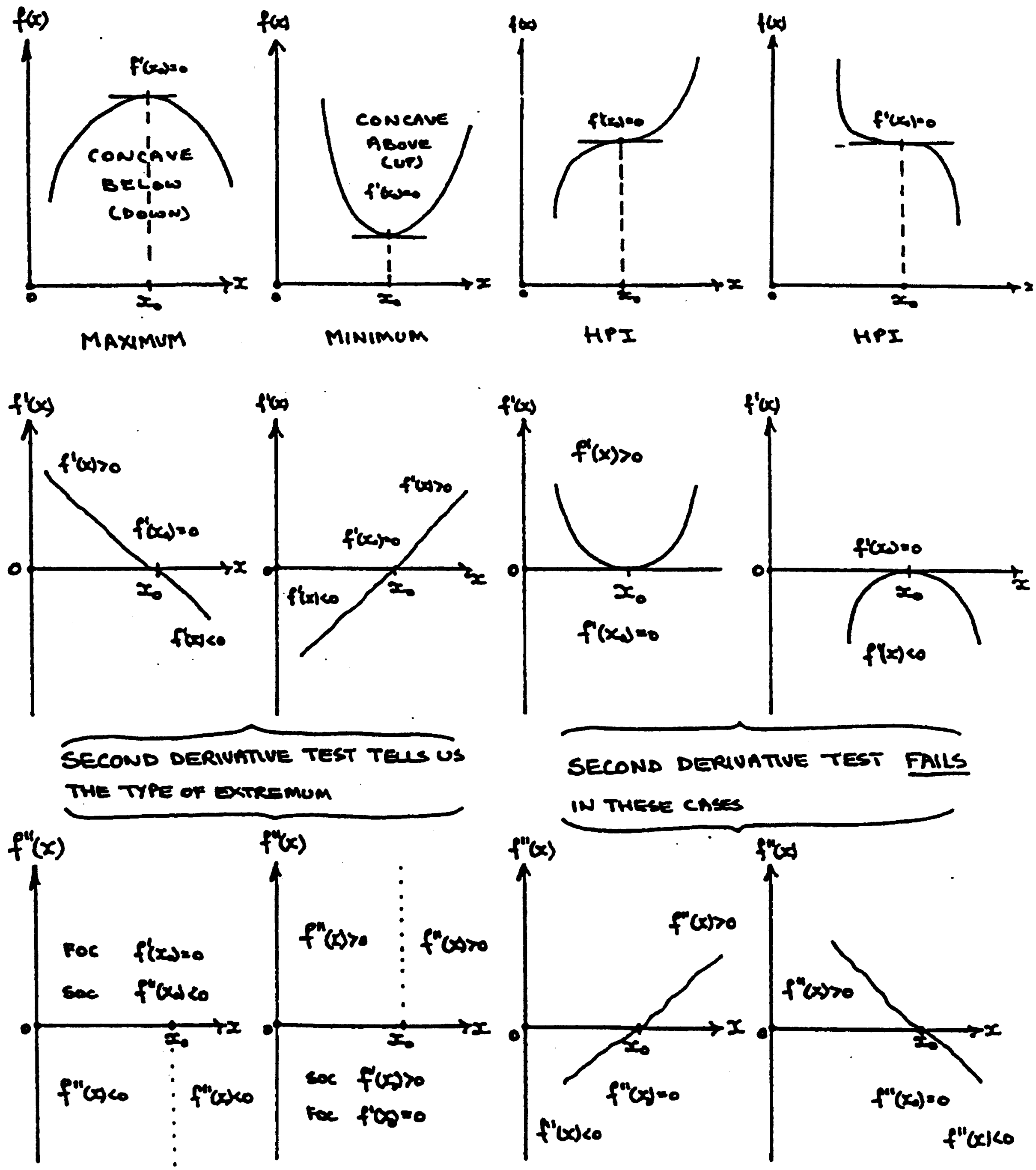

MAXIMUM
MINIMUM
HPI
HPI
CONCAVE BELOW (DOWN)
CONCAVE ABOVE (UP)
SECOND DERIVATIVE TEST TELLS US THE TYPE OF EXTREMUM
SECOND DERIVATIVE TEST FAILS IN THESE CASES
FOC
SOC

Figure 3

FOC: $\pi'(Q)=0 \Leftrightarrow MR = MC$ SOC: $\pi''(Q)<0 \Leftrightarrow$ slope MR < slope MC or MC cuts MR from below.

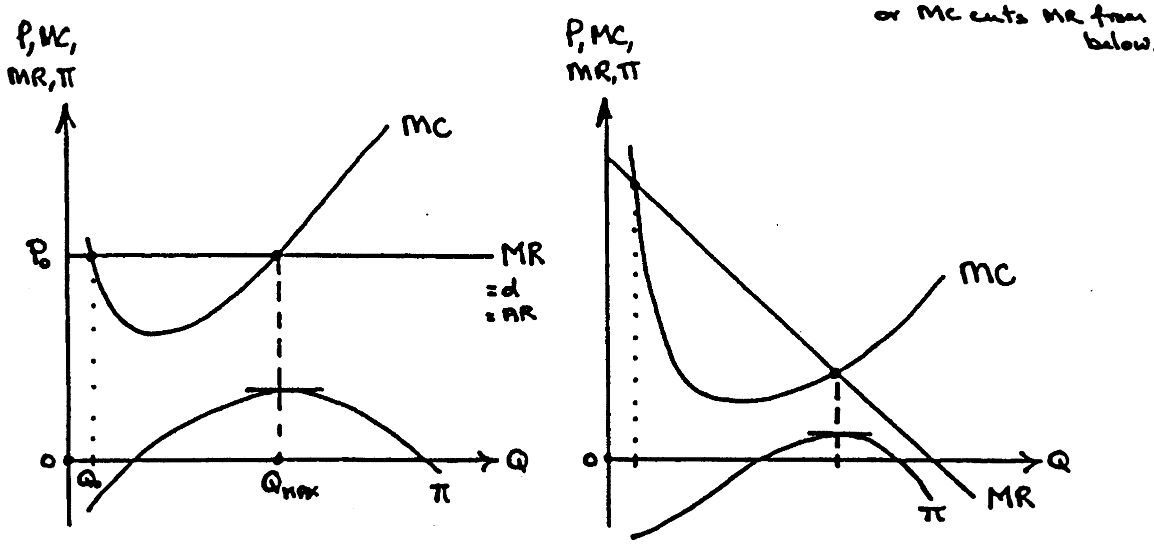
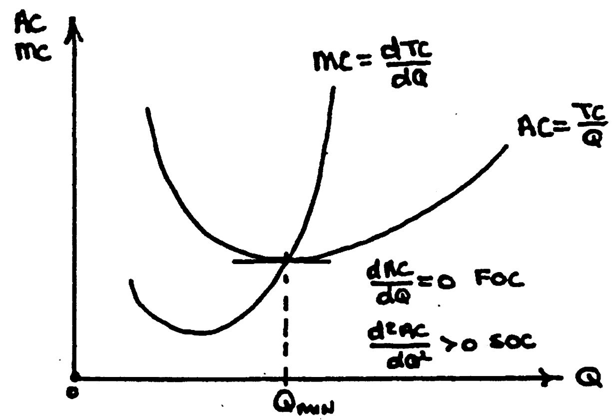


Figure 4



$$\frac{dAC}{dQ} = \frac{1}{Q}(MC - AC) = 0 \text{ at } Q_{MIN}, \text{ i.e. } MC = AC$$

$$\frac{d^2AC}{dQ^2} = \frac{1}{Q}\left(\frac{dMC}{dQ} - \frac{dAC}{dQ}\right) > 0 \text{ at } Q_{MIN}, \text{ i.e. } \underset{+}{\text{slope MC}} > \underset{0}{\text{slope AC}}$$

so MC cuts AC from below