MATHEMATICAL REVIEW 7

HIGHER-ORDER DERIVATIVES

1. In general the process of differentiation creates a new function (f' or dy/dx) -- called the **derived** function or **derivative** -- from an *original* function (f or y = f(x)), which is in turn differentiable (i.e., has a continuous graph with no sharp points). In general the derivative of the new function (f'' or d^2y/dx^2) will also be differentiable, and so on.

If $y = f(x) \quad (x \in R)$

then $\frac{dy}{dx} = f'(x) \quad (x \in R)$ FIRST ORDER

and $\frac{d^2y}{dx^2} = \frac{d(dy/dx)}{dx} = f''(x) \; (x \in R)$ SECOND ORDER

and $\frac{d^3y}{dx^3} = \frac{d(dy^2/dx)}{dx} = f'''(x) \; (x \in R)$ THIRD ORDER

and $\frac{d^ny}{dx^n} = \frac{d(\, dy^{n-1}/dx)}{dx} = f^n(x) \; (x \in R)$ Nth. ORDER

EXAMPLES:

Let $y = f(x) = 6x^5 \qquad (x \in R)$

then $\frac{dy}{dx} = f'(x) = 30x^4 \quad (x \in R)$

and $\frac{d^2y}{dx^2} = f''(x) = 120x^3 \qquad (x \in R)$

and $\frac{d^3y}{dx^3} = f'''(x) = 360x^2 \qquad (x \in R)$

and $\frac{d^4y}{dx^4} = f^{iv}(x) = 720x \qquad (x \in R)$

and $\frac{d^5y}{dx^5} = f^{v}(x) = 720 \qquad (x \in R)$

and $\frac{d^6y}{dx^6} = f^{vi}(x) = f^{vii}(x) = \ldots = \frac{d^ny}{dx^n} = f^n(x) = 0 \quad (x \in R).$

Note that the *fact that a derivative is zero does* ***not*** *mean that the derivative does not exist*.

Let $f: R^0 \rightarrow R^0$ s.t. $Q = f(L) = AL^{\alpha}$ $A, \alpha \in R$, $A > 0, 0 < \alpha < 1$

then $dQ/dL = A\,\alpha\,L^{\alpha-1} = \alpha\,A\,L^{\alpha-1} = \alpha\,A\,L^{\alpha}L^{-1} = \alpha\,A\,L^{\alpha} / L$

$= \alpha\,Q/L = \alpha\,AP_L = MP_L > 0.$

and $d^2Q/dL^2 = (\alpha-1)\,\alpha\,A\,L^{\alpha-2} = \alpha(\alpha-1)A\,L^{\alpha-2} = \alpha(\alpha-1)A\,L^{\alpha}L^{-2}$

$= \alpha(\alpha-1)\;Q/L^2 < 0.$

and $d^3Q/dL^3 = (\alpha-2)\,\alpha(\alpha-1)\,A\,L^{\alpha-3} = \alpha(\alpha-1)\,(\alpha-2)\,A\,L^{\alpha-3}$

$= \alpha(\alpha-1)\,(\alpha-2)\;Q/L^3 > 0.$

2. f(x) is the *height* of the graph of the *original* function f.

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f'(x) is the *height* of the graph of the *derived* (marginal) function f'.

f'(x) is the *slope* of the graph of the function f (it is the marginal function corresponding to the total function f).

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f''(x) is the *height* of the graph of the (second order) *derived* function f''.

f''(x) is the *slope* of the graph of the (first order/marginal) *derived* function f'.

f''(x) is the *curvature* (rate of change of the rate of change) of the *original* function f.

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f'''(x) is the *height* of the graph of the (third order) *derived* function f'''.

f'''(x) is the *slope* of the graph of the (second order) *derived* function f''.

f'''(x) is the *curvature* of the graph of the (first order/ marginal) *derived* function f'.

f'''(x) is the rate of change of the *curvature* of the *original* function f.

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EXAMPLE Let $y = f(x)$ $f(0) > 0$, $f'(x) > 0$, $f''(x) < 0$ $(x \in R)$

then f is a (monotonic) **increasing** function that is *increasing* at a *decreasing* rate and which cuts the vertical axis above the origin. Each image is *higher* than its predecessor is, but the *increment* in y is smaller each time.

Because $f'(x) > 0$ and $f''(x) < 0$ ($x \in R$) f' has positive images (since f is *increasing*) and f' has a negative slope (because f is increasing at a *decreasing* rate). We do not know if f' is decreasing at an increasing or decreasing rate (we do not know anything about the *curvature* of the graph of f') since we are not given the sign of the third order derivative. We say that this function, f, is **concave from below** because it has a negative second order derivative; i.e. the graph of the function will look like a hill not like a valley. (You should ask yourself what we could say about the graphs of functions that have first and second order derivatives that are permutations of the ones specified in the first sentence.)

Because $f''(x) < 0$ ($x \in R$) f'' has negative images because f' has a negative slope. We cannot say anything else about the graph of f'' if all we know is that $f'(x) < 0$.

(Figure 1 goes here.)

3. Beware of negatively sloped graphs! (See Figure 2.)

$f'(x_0) < f'(x_1)$ $f''(x) > 0$

hence f is **de**creasing at an *in*creasing *rate*. f is **convex** with regard to the origin: f is convex from below.

$f'(x_0) > f'(x_1)$ $f''(x) < 0$

hence f is **de**creasing at a **de**creasing rate.

4. In economics the phrases "**diminishing** marginal utility," "**diminishing** returns to the variable factor" alert us to functions with **negative** second derivatives; i.e.

$$\frac{d^2U}{dx^2} < 0 \text{ and } \frac{d^2Q}{dL^2} < 0, \text{ etc.}$$

5. Let $TC = F + V(Q)$ $\quad TC(0) = F, \; TC' = V'(Q) > 0$

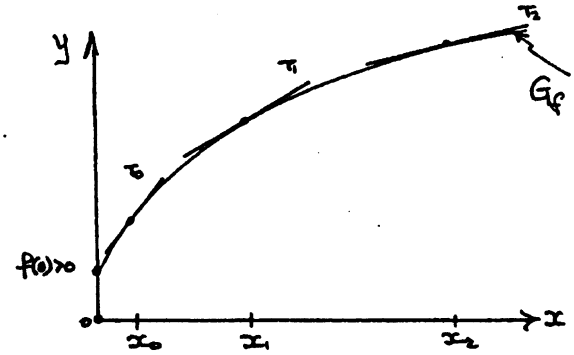
$$V''(Q) < 0 \quad 0 < Q \leq Q_0$$
$$V''(Q) > 0 \quad 0 > Q_0.$$

$$MC = \frac{dTC}{dQ} = V(Q) = \frac{dVC}{dQ}$$

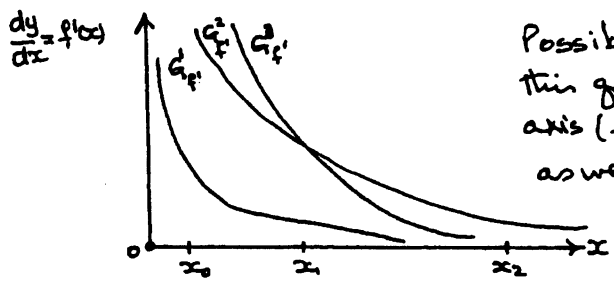
i.e., fixed costs do not effect MC (and hence do not effect profit maximizing price or output in the short run).

Then dMC/dQ (the **slope** of the MC curve and the *curvature* of the TC curve) is negative for outputs between 0 and Q_0, reaches a minimum at Q_0 -- the point of diminishing returns-- and is positive for outputs greater than Q_0. (See Figure 3.)

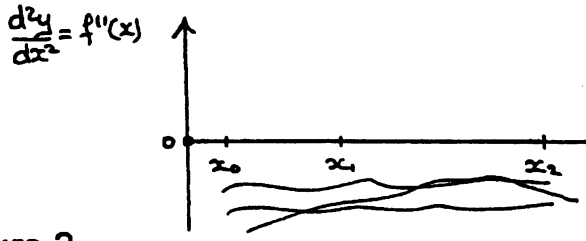
Figure 1.



G_f is a possible graph for a function f. Its slope and curvature are known qualitatively and it has a positive vertical intercept. It is a monotonically increasing function, increasing at a decreasing rate (concave down).



Possible graphs of f'. We know that this graph cannot cut the horizontal axis ($f'>0$) but that it flattens as we move to the right $f'(x_0)>f'(x_1)>f'(x_2)$.



We know the graph of f'' lies below the horizontal axis but know nothing about its slope or curvature

Possible graphs of f''.

Figure 2.

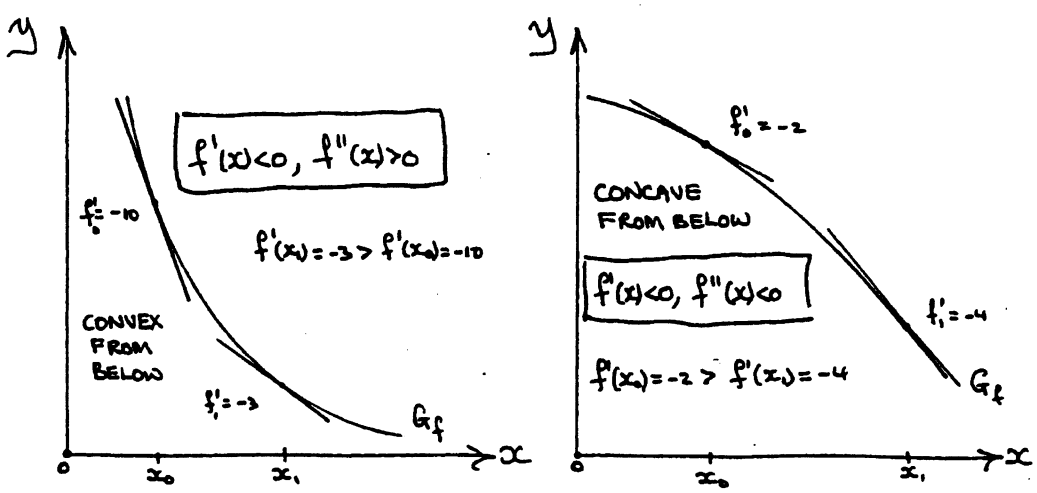
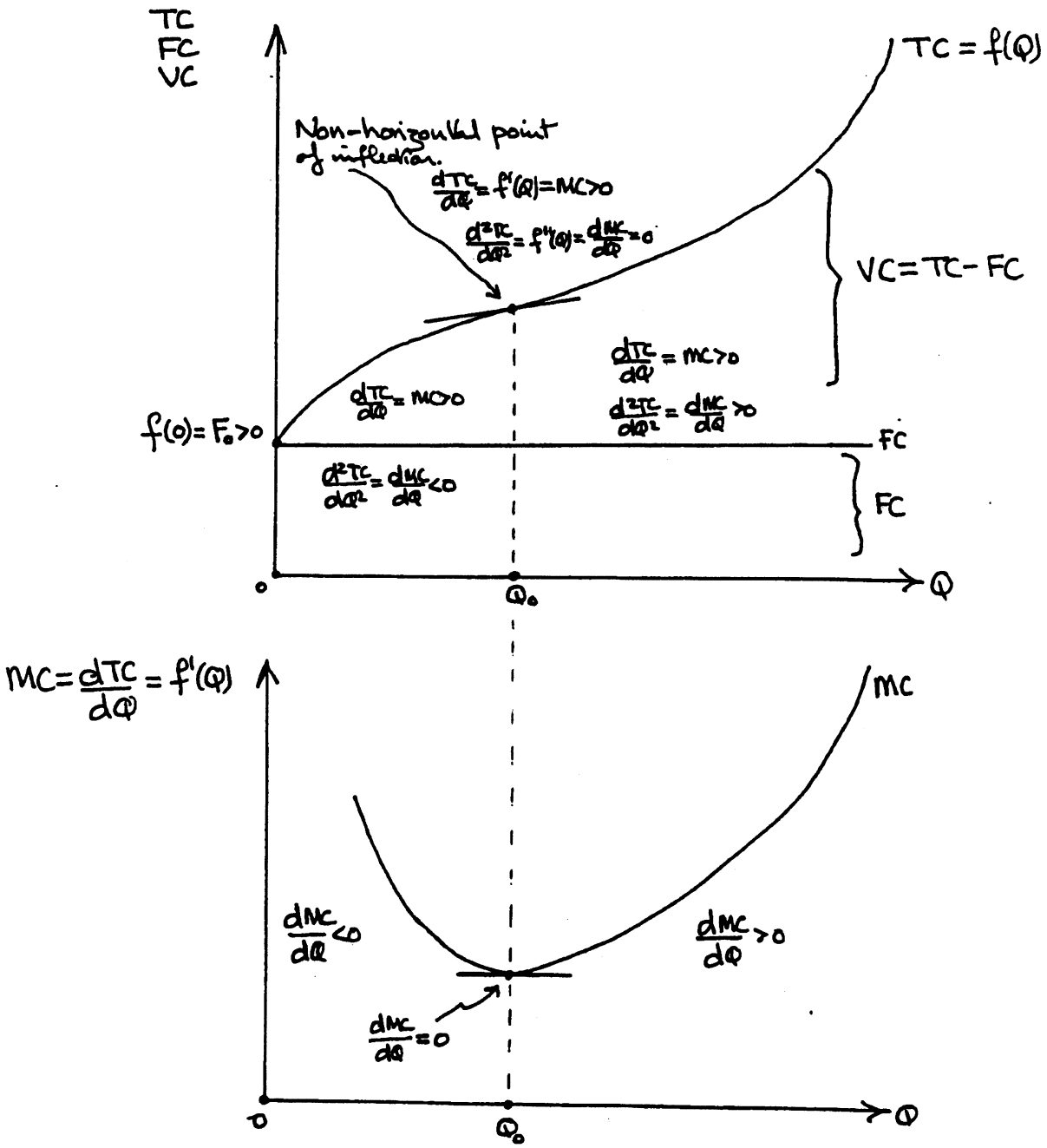


Figure 3.


TC
FC
VC
TC = f(Q)
Non-horizontal point of inflection.
dTC/dQ = f'(Q) = MC > 0
d²TC/dQ² = f''(Q) = dMC/dQ = 0
VC = TC − FC
dTC/dQ = MC > 0
d²TC/dQ² = dMC/dQ > 0
dTC/dQ = MC > 0
f(0) = F₀ > 0
FC
d²TC/dQ² = dMC/dQ < 0
FC
0
Q₀
Q
MC = dTC/dQ = f'(Q)
MC
dMC/dQ < 0
dMC/dQ > 0
dMC/dQ = 0
0
Q₀
Q