PART TWO

EXTREMIZATION MODELS USING UNIVARIATE CALCULUS

MATHEMATICAL REVIEW 5

INTRODUCTION TO DIFFERENTIAL CALCULUS

NOTE: *Chapter 4 of* ***A&L*** *covers all of the material on single variable differential calculus that you need to know for the 208 course.*

Because in economics we seldom have the luxury of knowing the specific algebraic forms of the functions we study, the calculus we do in ECON 208 does not require you to do much by the way of algebraic manipulation, *but* it does require a good **conceptual** grasp of the ideas underlying differential calculus.

Chapter 5 of A&L has some useful economic applications of the technique).

Those of you who are new to calculus will need to **study** this material intensively if you are to **master** it. **Because of the excellent coverage of the essential math in A&L I am going to concentrate on the *economic applications* in the lectures, assignments, handouts, and exams.**

1. In economics we often need to calculate the slopes of non-linear functions; e.g., to calculate mpc's for generalized consumption functions, to determine marginal cost for nonlinear

total and variable cost functions, or to determine the elasticity of demand of non-linear demand curves.

2. We will associate the **slope** of the non-linear **function** with the **slope** of the corresponding **tangent** line drawn to the curve at the relevant point. The slope of the tangent can be approximated by a **secant** drawn between the point of interest and some other point on the graph of the function. (See Figure 1.)

As we allow B to move along the graph towards A, the secant (with slope $\Delta y/\Delta x$) becomes a better and better approximation to the tangent, T_A, which at A, has the same slope as the graph of the function; i.e., dy/dx. In other words as x_B approaches x_A the approximation $\Delta y/\Delta x$ gets closer and closer to the true value dy/dx. The set of values of the secant approximations is a sequence of real numbers, such as 4, 4.12, 4.194, 4.203, 4.4, ..., 4.487, 4.488, 4.921, 4.934, 4.956, 4.971, 4.989, 4.991, 4.998, 4.999, 4.9995, 4.9998, 4.9999, ... etc. It *appears* that these numbers are getting closer and closer to the number 5 as B is getting closer and closer to A. Equivalently, we can say that as h = Δx gets closer and closer to zero it *appears* that the secant approximation is getting closer and closer to 5. It is the job of the mathematician to **prove** this *assertion.* **If** such a **proof** is forthcoming (it involves fearsome things called epsilons and deltas which you do not have to worry yourselves about -- mathematicians have been doing these proofs for one hundred and fifty years and so we can assume that they have got them right!) we then say that the sequence of secant approximations possesses a *limit* (the **number** 5), or that $\Delta y/\Delta x = [f(x+h)-f(x)]/h$ approaches 5 as Δx=h approaches zero (they may never *actually take* the value zero). **If** this limit exists then we say that f is *differentiable* at x = x_A and that the *derivative* of f, evaluated at x_A, is equal to that limit (in our case 5). If the function has a derivative at **each** point in its domain then we say that it is a *differentiable* function. In ECON 208 (and in most of your upper division courses!) we will **assume** that the functions we encounter are differentiable functions (actually all we usually require is that the function possesses continuous first and second order derivatives; third order derivatives if we are interested in the curvature of the graph of the marginal function).

3. **DEFINITION**

If $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ **exists**

then we call this value the **derivative** of the function which we denote by dy/dx = f'(x);

i.e., $$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

$$= \lim_{x \to 0} \frac{\Delta y}{\Delta x}$$ **if** that limit **exists.**

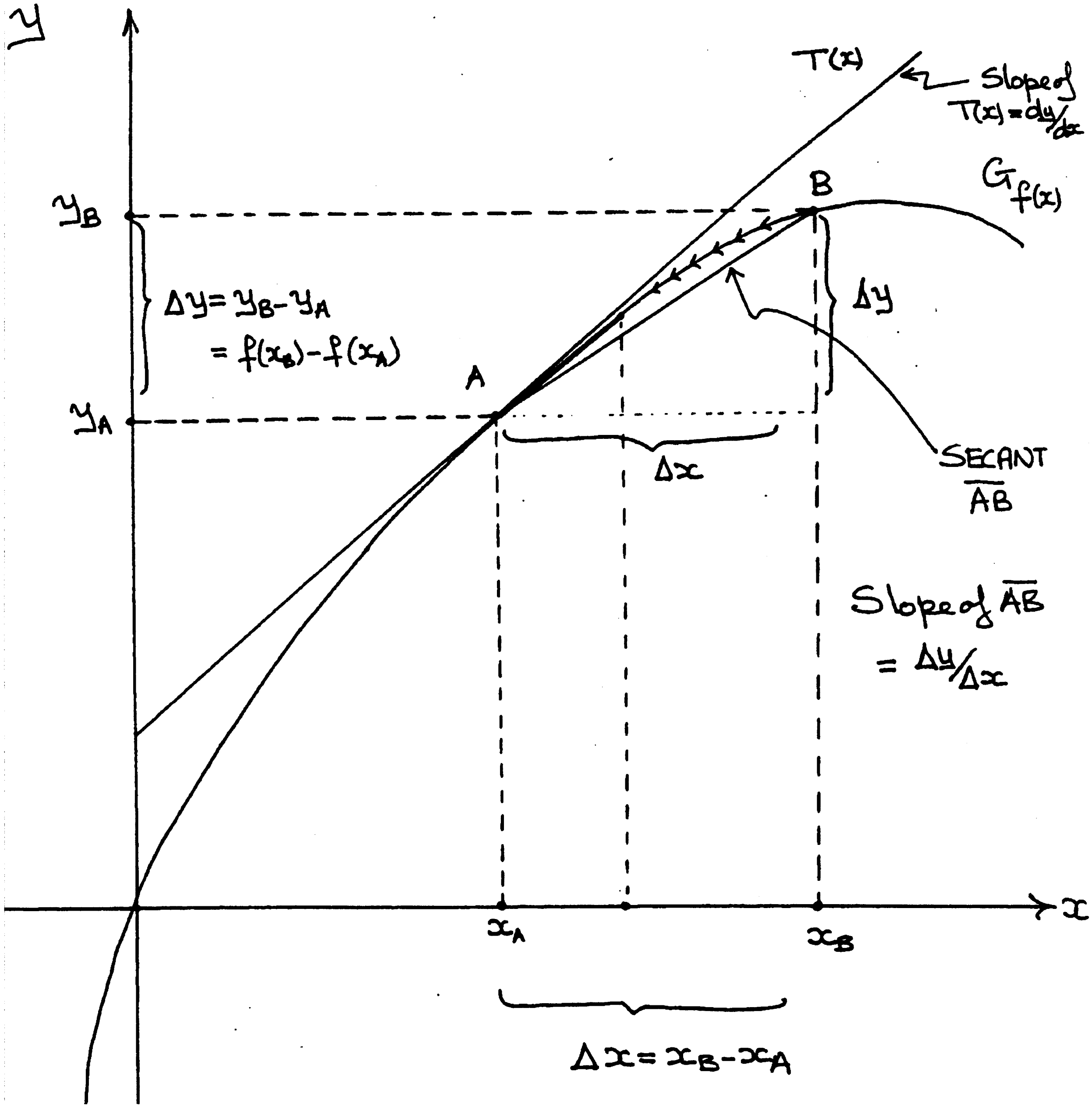
The derivative is a derived **function** which is brought out by the notation f'(x) ($x \in D_f$ where D_f is usually some subset of the set of real numbers). The **height** of the f'(x) graph -- the value of dy/dx at a particular value of x in the domain of the function -- is equal to the **slope** of the graph of the original function, f. Since the slope is *unique* and defined at *every* x for which f is differentiable, f' is a function in its own right (and in economics contexts is usually assumed to be at least twice differentiable itself, which we indicate by writing $f \in C^2$).

In economics all **marginals** are derivatives: e.g. *marginal cost* is the rate of change of total cost with respect to output and hence is a measure of the slope of the graph of the total cost function; *marginal revenue* is the rate of change of total revenue with respect to output and hence is a measure of the slope of the graph of the total revenue function; *marginal product* is the rate of change of total product (output) with respect to changes in labor (or capital) input and hence is a measure of the slope of the graph of the total product function; the *marginal propensity to consume* is the rate of change of consumption with respect to income and is therefore a measure of the slope of the graph of the consumption function.

Differentiation is a technique in which: (a) we **approximate** one function, f, by a simpler function, T(x) -- the **linear** tangent function; and (b), *strictly speaking,* all changes are assumed to be *"infinitesimally" small* (to ensure that the approximation is a "good" one). Differentiation is therefore a "myopic" technique, but nonetheless immensely powerful when used judiciously.

It is very important that you keep two *intuitive concepts* of the derivative in mind. We can think of derivatives as measuring the **slopes** of the graphs of the function in question; and we can think of the derivative as a measure of **the rate of change** of the dependent variable of the function with respect to ("infinitesimal") changes in the independent variable (or, sometimes, a parameter).

Figure1


y
x
T(x)
Slope of T(x) = dy/dx
B
$G_{f(x)}$
y_B
y_A
$\Delta y = y_B - y_A$
$= f(x_B) - f(x_A)$
A
Δy
Δx
SECANT $\overline{AB}$
Slope of $\overline{AB}$ = $\Delta y / \Delta x$
x_A
x_B
$\Delta x = x_B - x_A$

MATHEMATICAL REVIEW 6

BASIC RULES OF DIFFERENTIAL CALCULUS

1. (ADDITIVE) CONSTANT RULE

If $y = f(x) = c \quad (x \in R), \ (c \in R)$ then $\frac{dy}{dx} = f'(x) = 0 \quad (x \in R)$.

Let $y = f(x) = 4$ then $\frac{dy}{dx} = f'(x) = 0$.

Let $I = I(Y) = I_o$ then $\frac{dI}{dY} = 0$.

Let $FC = FC(Q) = F_o$ then $\frac{dFC}{dQ} = 0$.

2. IDENTITY FUNCTION RULE

If $y = f(x) = x \quad (x \in R)$ then $\frac{dy}{dx} = f'(x) = 1 \quad (x \in R)$.

Let $AS = g(Y) = Y$ then $\frac{dAS}{dY} = 1$.

Let $Q = f(Q) = Q$ then $\frac{dQ}{dQ} = 1$.

3. **MULTIPLICATIVE CONSTANT RULE FOR IDENTITY FUNCTIONS**

If $y = g(x) = b\, f(x)$ $(x, b \in R)$, where f is an identity function

then $\dfrac{dy}{dx} = g'(x) = b\, f'(x) = b$ $(x \in R)$, i.e. $y = b\, f'(x) = b.1 = b.$

Let $TR = R(Q) = P_0\, Q$ (perfect competition)

then $\dfrac{dTR}{dQ} = MR = R'(Q) = P_{0.}$

Let $W = f(L) = w_0\, L$ (perfect competition in the labor market)

then $\dfrac{dW}{dL} = f'(L) = w_0$ where W is the “wage bill” and w_0 is

the real wage.

4. **POWER FUNCTION RULE**

If $y = f(x) = x^n$ $(x \in R)$, $(n \in R)$

then $\dfrac{dy}{dx} = f'(x) = nx^{n-1}$ $(x \in R)$.

Let $y = x^2$ then $dy/dx = 2x$

Let $y = x^{1/2}$ then $dy/dx = ½\, x^{-1/2}$

Let $y = x^0 = 1$ then $dy/dx = 0.x^{0-1} = 0$

Let $y = x^{47}$ then $dy/dx = 47x^{46}$

5. **GENERAL MULTIPLICATIVE CONSTANT RULE**

> If $y = f(x) = b\,x^n \qquad (x \in R), \ (b,n \in R)$
>
> then $\dfrac{dy}{dx} = f'(x) = bnx^{n-1} \qquad (x \in R).$

Let $\quad TC = TC(Q) = VC(Q) = bQ^3 \ \ (Q \in R^0)$

then $\dfrac{dTC}{dQ} = \dfrac{dVC}{dQ} = 3bQ^2.$

Let $Q = f(L) = AL^{\alpha} \ \ A>0,\ 0<\alpha<1\ (L \in R^0)$ where $A, \alpha \in R$,

then $dQ/dL = f'(L) = A\alpha\, L^{\alpha-1} = \alpha\, A\, L^{\alpha-1} = \alpha\,(A\,L^{\alpha})\,L^{-1} = \alpha\, Q/L$,

i.e. $MP_L = \alpha\, AP_L$.

6. **SUM and DIFFERENCE RULES**

> (a) If $y = f(x)$ and $z = g(x)$ and $w = y + z = h(x) = f(x) + g(x)$
>
> then $\dfrac{dw}{dx} = \dfrac{dy}{dx} + \dfrac{dz}{dx} \qquad$ or $\qquad \dfrac{d\overset{w}{\cancel{y}}}{dx} = h'(x) = f'(x) + g'(x).$

Let $\quad y = 3 + 2x + 4x^2$

then $\dfrac{dy}{dx} = 0 + 2 + 8x = 2 + 8x.$

Let $\quad AD = C(Y) + I_0 + G_0 + X_0$

then $\dfrac{dAD}{dY} = C'(Y) + 0 + 0 + 0 = C'(Y).$

Let $\quad TC = TC(Q) = FC(Q) + VC(Q) = F_0 + VC(Q)$

then $\dfrac{dTC}{dQ} = 0 + VC'(Q) = VC'(Q) = MC.$

(b) If $y = f(x)$ and $z = g(x)$ and $w = h(x) = f(x) - g(x) = y - z$

then $\dfrac{dw}{dx} = \dfrac{dy}{dx} - \dfrac{dz}{dx}$ or $\dfrac{dw}{dx} = h'(x) = f'(x) - g'(x).$

If $y = f(x) = 3x^2 - 4x^3$

then $f'(x) = dy/dx = 6x - 12x^2$

If $\quad \Pi(Q) = R(Q) - C(Q)$

then $\Pi'(Q) = R'(Q) - C'(Q) = MR - MC.$

If $\quad AD(Y) = C(Y) + I(Y) + G(Y) + X(Y) - M(Y)$

then $AD'(Y) = C'(Y) + I'(Y) + G'(Y) + X'(Y) - M'(Y)$

$$= C'(Y) + 0 + 0 + 0 - M'(Y)$$

$$= C'(Y) - M'(Y)$$

$$= c(1\text{-}t)Y - m(1\text{-}t)Y > 0.$$

Figure 1

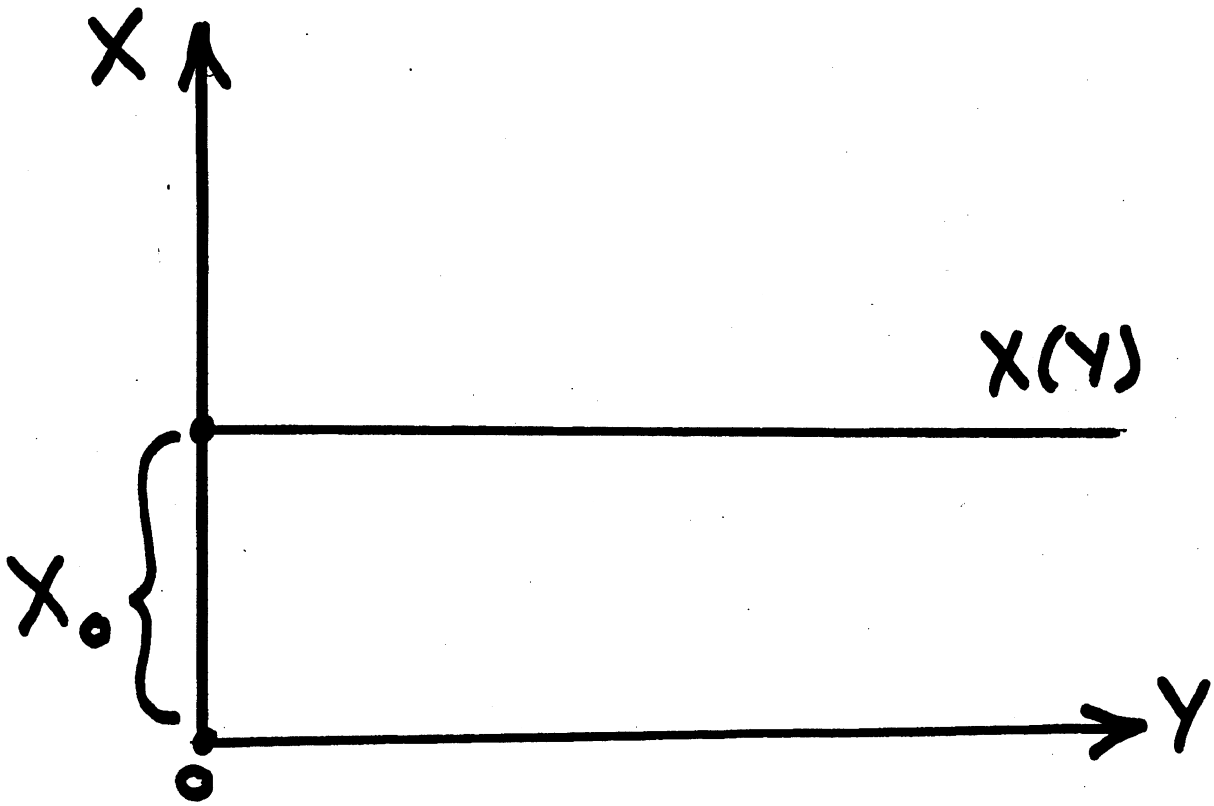
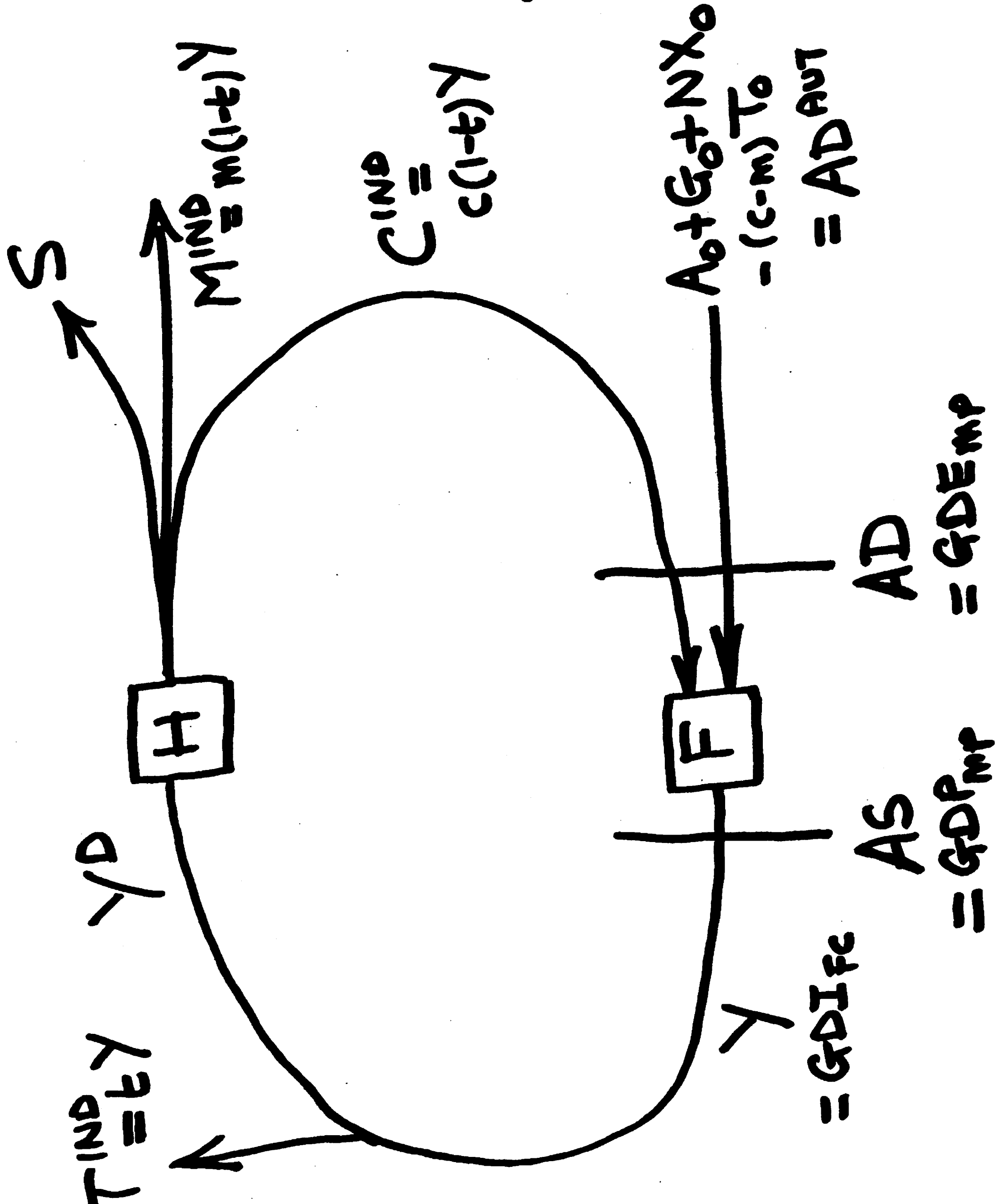


Figure 2



S
$M^{IND} = m(1-t)Y$
$C^{IND} = c(1-t)Y$
$A_0 + G_0 + NX_0 - (c-m)T_0 = AD^{AUT}$
H
F
$AD = GDE_{mp}$
$AS = GDP_{mp}$
$Y = GDI_{fc}$
Y^D
$T^{IND} = tY$

8

7. **PRODUCT RULE**

If $y = f(x)$ and $z = g(x)$ and $w = h(x) = f(x)g(x) = yz$

then $\dfrac{dw}{dx} = \dfrac{dy}{dx} z + y \dfrac{dz}{dx}$ or $h'(x) = f'(x)g(x) + f(x)g'(x)$.

Let $y = (2x - 1)(3x^2 + 2)$

then $\dfrac{dy}{dx} = (2)(3x^2 + 2) + (2x - 1)(6x)$

$= 6x^2 + 4 + 12x^2 - 6x$

$= 18x^2 - 6x + 4$

$= 2(9x^2 - 3x + 2)$.

Let $TR = f(Q) = P(Q)Q$

then $MR = \dfrac{dR}{dQ} = \dfrac{dP}{dQ} Q + P \dfrac{dQ}{dQ} = \dfrac{dP}{dQ} Q + P$,

hence $MR = \dfrac{dP}{dQ} . Q . \dfrac{P}{P} + P . \dfrac{P}{P} = \dfrac{dP}{dQ} . \dfrac{Q}{P} . P + P$.

But $PED = \dfrac{dQ}{dP} . \dfrac{P}{Q}$ and so $\dfrac{dP}{dQ} . \dfrac{Q}{P} = 1/PED$,

hence **$MR = P(1 + 1/ PED)$**.

(Show that $MR = P_0$ under the special case of perfect competition.
Show that P is **always** equal to AR.)

8. **QUOTIENT RULE**

If $y = f(x)$ and $z = g(x)$ and $w = h(x) = f(x)/g(x) = y/z$ $z = g(x) \neq 0$

then $\dfrac{dw}{dx} = \dfrac{\dfrac{dy}{dx} z - y \dfrac{dz}{dx}}{z^2}$ or $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$.

Let $y = 1$ and $z = x$ then $w = 1/x$ and so

$$\frac{dw}{dx} = \frac{\dfrac{d1}{dx} . x - 1. \dfrac{dx}{dx}}{[x]^2}$$

$$= \frac{1}{x^2}.$$

Let $y = (2x + 4)/(x - 2)$ $(x \in R \text{ and } x \neq 2)$

then $\dfrac{dy}{dx} = \dfrac{(2)(x - 2) - (2x + 4)(1)}{(x - 2)^2} = \dfrac{2x - 4 - 2x - 4}{(x - 2)^2} = 0.$

Let $TC = C(Q)$, $C'(Q) > 0$, $C(0) = F_o$ and $(Q \in R^0)$

Then $AC = \dfrac{TC}{Q} = \dfrac{C(Q)}{Q}$ $(Q \in R^+)$ [i.e., $Q \neq 0$!]

and so $\dfrac{dAC}{dQ} = \dfrac{C'(Q) . Q - C(Q) . 1}{Q^2}$ where $C'(Q) = MC$

$$= \frac{C'(Q)}{Q} - \frac{C(Q)}{Q} . \frac{1}{Q} \text{ (where } \frac{C(Q)}{Q} = AC\text{),}$$

hence $\frac{dAC}{dQ} = \frac{1}{Q} [C'(Q) - \frac{C(Q)}{Q}] = \frac{1}{Q} [MC - AC]$.

Note that this is a perfectly general result and could be proved for $y = f(x)$ and $\frac{d(y/x)}{dx}$.

You should show that this result holds for the AR, AVC, AP_L, AP_K, and apc functions as well as your GPA.

MATHEMATICAL REVIEW [OPTIONAL]**

CHAIN OR FUNCTION-OF-A FUNCTION OR COMPOSITE FUNCTION RULE

If $z = g(y)$ and $y = f(x)$ $(x \in R)$

then $z = h(x) = g[f(x)]$ $h = g \circ f: x \to g[f(x)]$ (we say h is a *function of a function* or a *composite function*) and

$$h'(x) = g'[f(x)] \cdot f'(x) \quad (x \in R)$$

or $$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}.$$

Let $z = 2y^2$ and $y = 3x$ $(x \in R)$

then $\frac{dz}{dx} = (4y)(3) = 12y = 12(3x) = 36x.$

Remember we are differentiating with regard to x and therefore we need to rewrite 12y into a form just involving x.

Let $z = 4y^3 + 2y$ and $y = 2x + 1$ $(x \in R)$

then $\frac{dz}{dx} = (12y^2 + 2)(2) = 24y^2 + 4 = 24(2x + 1)^2 + 4$ $(x \in R)$.

Let $Q = Q(L)$ $(L \in R^0)$ and $Q'(L) = \frac{dQ}{dL} = MP_L$ and $P = P(Q) = AR$

Let $y = \sqrt{(bx + cx^4)^3} = (b + cx^4)^{3/2}$

then $\frac{dy}{dx} = (\frac{3}{2}(bx + cx^4)^{1/2})\,(4cx^3)$.

Let $y = (a + bx^2)^{1/2} = \sqrt{(a + bx^2)}$

then $\frac{dy}{dx} = (\frac{1}{2}(a + bx^2)^{-1/2}\,(2xb) = \frac{bx}{\sqrt{(a + bx^2)}}$.

Let $y = u^{½}$ where $u = f(x)$ $(x \in R)$ $(x \in R,\ x \geq 0)$

then $\frac{dy}{dx} = \frac{1}{2}u^{-½}.\ f'(x) = \frac{1}{2}(\frac{1}{u^{½}})\,f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$.

Let $Y^e = \frac{1}{1-c}\ A_0$ where $A = C_0 + I_0 > 0$ and $0 < c < 1$ $(Y^e \in R^0)$

then $\frac{dY^e}{dC} = \frac{dY^e}{dA}\,.\,\frac{dA}{dC} = \frac{1}{1-c}\,.\ 1 = \frac{1}{1-c} = k > 1$ (a *shift* or *intercept* multiplier).

Alternatively if we write $z = 1\text{-}c$ then $Y^e = z^{-1} A$ and

$$\frac{dY^e}{dc} = (-z^{-2} A)(-1) = (\frac{-1}{(1-c)^2} A)(-1) = \frac{A}{(1-c)^2} = (\frac{1}{1-c})\,(\frac{1}{1-c}.A) = \frac{dY^e}{dC_0}\,Y_0^e$$

(a *pivot* multiplier).

$(Q \in R^0)$ -- the AR function which is the inverse function of the (1:1) demand function. Then TR = P(Q)Q = P[Q(L)]Q(L) = f(L)

and so $\frac{dTR}{dL} = \frac{dTR}{dQ} \cdot \frac{dQ}{dL}$

i.e., $\frac{dTR}{dL} = MRP_L = \frac{d\{P[Q(L)] \cdot Q(L)\}}{dL}$

$$= \frac{dP[Q(L)]}{dL} Q(L) + P[Q(L)] \frac{dQ(L)}{dL}$$

$$= \frac{dP}{dQ} \cdot \frac{dQ}{dL} \cdot Q + P. \frac{dQ}{dL}$$

$$= [\frac{dP}{dQ} \cdot Q + P]. \frac{dQ}{dL}$$

$$= MR \cdot MP_L \; .$$

A special case of the Chain rule arises when $y = g[f(x)] = f(x)]^n$, i.e., the g function is a power function, then

$\frac{dy}{dx} = g'[f(x)]f'(x) = n[f(x)]^{n-1} \cdot f'(x).$

Let $y = (a + bx + cx^2)^4$

then $\frac{dy}{dx} = (4(a + bx + cx^2)^3)\,(b + 2cx).$

ECONOMIC THEORY 11

ELASTICITY

INTRODUCTION

1. Economists want to answer questions such as: how much does Y change when X changes? That is, we would like to be able to say something about the size of ΔY brought about by a change in ΔX. This would seem to take us out of the qualitative world in which we have been operating until now and into the world of quantitative analysis. However, as we shall see, there are qualitative things that can be said about this topic.

2. At first sight it would appear that we could use the slope of the graph of the function relating Y to X to answer our question since the slope coefficient is $\Delta Y/\Delta X$, but unfortunately **the slope coefficient is not a pure number because its magnitude depends on the units in which Y and X are measured.** (Since you can now do simple differential calculus we will use our derivative notation to indicate the slope of the graph of the function at a point on that graph, i.e. we will use dY/dX rather than $\Delta Y/\Delta X$, although for *linear* functions the two ways of measuring the slope yield the same answer.)

   This point about units of measurement may be more obvious if we ask a slightly different question: which is more responsive to changes in price – the quantity demanded of Cadillac Sevilles or the quantity demanded of table salt? Specifically, say that we know that the price of Cadillac Sevilles fell by $3,000 last year and that sales increased by