MATHEMATICAL REVIEW 3

LINEAR FUNCTIONS

1. The simplest class of functions is the set of **CONSTANT** functions. A *constant* function has the form:

$$f: R \rightarrow R$$

$$\text{s.t. } y = f(x) = a.$$

Where **a** is a real parameter (a *parameter* is a *constant* to which we may assign a value), or

$$y = f(x) = a \quad (x \in R) \quad (a \in R).$$

[Figure 1 goes here.]

Economic examples of constant functions arise naturally in elementary macroeconomics where we often specify that a variable is *exogenous* (*autonomous*) which means that it does not vary with, and is not affected by, any other variable in the system (in macroeconomics this usually means - does not vary with *income*, Y); e.g., we might make government expenditure exogenous, writing $G = G_0$ or, more formally, $G = G(Y) = G_0$ $(Y \in R^0)$ $G_0 > 0$.

Another obvious economic example is the fixed cost function $FC = F(Q) = F_0$ $(Q \in R^0)$ where F_0 is a positive real constant.

2. The simplest class of functions that allow y to *vary* with x is the set of **IDENTITY** functions. An *identity* function has the form:

$$f: R \to R$$

$$\text{s.t. } y = f(x) = x$$

or

$$y = f(x) = x \quad (x \in R).$$

Perhaps the best example of an identity function in economics is the Aggregate Supply curve in the simple Income Determination model, where we have

$$AS = f(Y) = Y \quad (Y \in R^0)$$

which yields the familiar 45^0 graph of the so-called "Keynesian Cross" diagram.

It should be remembered that **every variable is a function of itself** so that $Q = f(Q) = Q$, and $P = f(P) = P$, and $Y = f(Y) = Y$ etc., a fact we will use frequently below.

[Figure 2 goes here].

3. The next simplest set of functions are those in which y is some multiple of x, which I will call **MULTIPLICATIVE** functions. These functions take the form:

$$f: R \to R$$

$$\text{s.t. } y = f(x) = bx \quad b \in R, b \neq 0,$$

where *b*, called the *multiplicative* constant, can be positive or negative but not zero.

Economic examples of multiplicative functions include the Total Revenue function under perfect competition

$$f: R^0 \to R^0$$

$$\text{s.t. } TR = f(Q) = P_0Q \qquad P_0 \in R^+ ,$$

and the long run Consumption Function

$$f: R^0 \to R^0$$

$$\text{s.t. } C = f(Y) = cY \qquad 0 < c < 1.$$

Where the symbol c is the long run marginal propensity to consume, written as mpc.

[Figure 3 goes here].

4. For the next few lectures we will concentrate on the set of **LINEAR** functions. These functions, at least in the simultaneous equation form, allow you to do a great deal of undergraduate economics algebraically. (Mathematicians call identity functions and multiplicative functions *linear functions* and what economists call linear functions mathematicians refer to as *affine functions*. We will adhere to the economists' terminology.) A *linear* function is of the form:

$$f: R \to R$$

$$\text{s.t. } y = f(x) = a + bx \qquad a, b \in R,\ a, b \neq 0.$$

This version of the linear function is said to be in ***slope-intercept form***, since *a* represents the *vertical intercept* of the graph of the function (if $y = a + bx$ and $x = 0$ then $y = a$ so that the vertical intercept is a), and *b* represents the slope of the graph of the function (where *slope* is defined to be rise over run: the vertical displacement divided by the horizontal displacement, i.e.$\Delta y/\Delta x$).

[Figure 4 goes here.]

If $y = a + bx$ and $y = 0$ then $x = -a/b$ (the *horizontal intercept* is $-a/b$).

So in $y = f(x) = a + bx$ **a** measures the *vertical intercept* and **b** measures the *slope* of the function. (What would the graphs look like if $(a > 0, b = 0)$, $(a = 0, b > 0)$, $(a < 0, b > 0)$, $(a > 0, b < 0)$, $(a = 0, b = 0)$, $(a < 0, b < 0)$)?

Note that the equation (relation) $x = c$ $(c, x, y \in R)$ is **not** a function. The graph of the relation is a *vertical* straight line. Note that this line has *no slope* since the statement "the slope of the vertical line is infinite" is strictly meaningless from a mathematical point of view because *infinity* is not a number, and the "run" in this case would be zero, and division by zero is not a defined mathematical operation.

Economists often specify consumption, demand, and supply functions in linear form in order to simplify the mathematics without losing much in terms of analytical sophistication or, over the relevant range of the variables, much in terms of empirical accuracy.

$$C = C(Y) = C_O + cY \quad (Y \in R^0) \quad C_O \geq 0, \quad 0 < c < 1.$$

Where $c = \Delta C/\Delta Y$ = mpc = the marginal propensity to consume.

We can use the Consumption Function to derive a Saving Function

$$S = Y - C = -C_0 + (1-c)Y.$$

Since Y = C + S, i.e. income is either consumed or saved, we can **prove** that the mpc and the mps sum to one:

$$\Delta Y = \Delta C + \Delta S$$

$$\frac{\Delta Y}{\Delta Y} = \frac{\Delta C}{\Delta Y} + \frac{\Delta S}{\Delta Y}$$

$$1 = \text{mpc} + \text{mps}.$$

(Could C_0 be negative? What *economic* distinction is brought about by setting $C_0 > 0$ or $C_0 = 0$?)

Economists believe that **the quantity demanded** of some commodity (good or service), Q^d, in constant quality units per unit of time, **depends on** the (relative) **price** of that commodity (among other things). We could therefore write $Q^d = f(P)$ $(P \in R^0)$ in general, and $Q^d = a + bP$ $(P \in R^0)$ as a special case. In this formulation Q^d is the **dependent** variable (*to be explained*) and P the **independent** variable (*doing the explaining*). The **causation** therefore goes **from** P **to** Q^d, ***not*** from Q^d to P.

$$f\colon P \to Q^d \qquad (P \in R^0)$$

where P is the set of all possible relative prices of the commodity and Q^d is the set of corresponding quantities demanded. (Notice that P and Q are both *endogenous* variables in the supply and demand *model*, although Q is a *dependent* variable in the demand *function* and P is an *independent* variable in the demand *function*.)

Therefore if our graphs are to correspond to the algebra we must put P on the horizontal axis and Q^d on the vertical axis (reversing the traditional Marshallian diagram).

YOU MUST BE ABSOLUTELY SURE THAT YOU UNDERSTAND THIS ARGUMENT SINCE WE SHALL CONTINUE TO APPLY IT TO ANY PIECE OF ECONOMICS THAT WE FORMULATE MATHEMATICALLY. If you are at all uncertain about the logic of what we are doing you must *ask me* to attempt to clarify it for you.

[Figure 5 goes here.]

$$Q^d = a + bP \quad (P \in R^0) \quad a > 0, b < 0, \quad a, b \in R.$$

(What happens to the demand curve if **a** is increased/ decreased or if **b** is increased/decreased? What is the economic interpretation of these changes?)

A linear supply curve can be written in the form:

$$Q^S = g(P) = c + dP \quad (P \in R^0) \quad c < 0, d > 0, \quad c, d \in R.$$

That is, Q^S is an **increasing** function of P, with a *negative* vertical intercept. (Why do we make c negative?) If P=0 then Q^S = c is the *vertical* intercept, and if Q^S = 0 then P = -c/d is the *horizontal* intercept of the supply curve.

[Figure 6 goes here.]

The technical issue which seems to still be troubling a lot of you is why we must draw our supply and demand diagrams with Q on the vertical axis and P on the horizontal axis **whenever we are doing a piece of mathematics** – whether that be number crunching, algebra, or calculus. Only when we are mathematically modeling *firms* will we revert to the familiar Marshallian coordinates.

In most cases you are having problems because you have not really mastered **the concept of a function and its associated graph**. A function consists of a domain set (from which the **independent** variable values are drawn), and a functional rule (in our case an algebraic equation) that tells us what to do to the x value we have drawn in order to find its image (the value of the **dependent** variable, y) in the codomain set.

Say we have the simple system given by:

$$f,g: R^0 \to R^0$$

$$Q^d = f(P) = 100 - 2P$$

$$Q^s = f(P) = -20 + 4P$$

$$Q^d = Q^s = Q^e.$$

You should by now be able to determine that $P^e = 20$ and $Q^e = 60$.

How do we plot the graphs of the demand and supply functions? The functional specification in line one tells us that Q and P are non-negative reals so that we may confine our attention to the first quadrant. The demand equation makes it clear that – as we would expect from our economic theory – **P** is the **independent** variable and that $\mathbf{Q^d}$ is the **dependent** variable. So the equation says that when we plot the graph of the demand function that it must have **P** on the **horizontal** axis and $\mathbf{Q^d}$ on the **vertical** axis. Further we know that the **vertical intercept** of the demand curve is **100** since f(0) = 100, and that the demand curve has a **slope** of **–2**. If you carefully plot the two curves you will find that they do indeed intersect in the first quadrant at the point with coordinates (20,60).

Now some of you want to plot the demand curves with P on the vertical axis and Q on the horizontal axis – the diagram you learned in 206 and which we often use (**whenever we are not doing mathematics**) in 208 too. HOW DO YOU PROPOSE TO DO THIS? Most of you do not seem to pursue this issue to its logical conclusions. What I think some of you wish to do is to draw a diagram in which P appears on the vertical axis and Q on the horizontal axis and in which the vertical intercept of the demand curve is still 100 and its slope is –2, and with a supply curve that has a vertical intercept of –20 and a slope of 4. If you plot such a diagram then you will discover that the curves intersect in the first quadrant but that the point of intersection has coordinates (20,60) which means that your **diagram** makes $Q^e = 20$ and $P^e = 60$ which does **not** agree with your algebraic solution! **Since your algebra is correct we must conclude that you have drawn the wrong diagram!**

An alternative approach that may be favored by some of you is to plot the **inverse** demand and supply functions, f^{-1} and g^{-1}, but then you are actually solving the algebraic model given by:

$$f^{-1}, g^{-1} : R^0 \to R^0$$

$$P^d = f(Q) = 100 - 2Q$$

$$p^s = f(Q) = -20 + 4Q$$

$$P^d = P^s = P^e.$$

The problem with this approach is that it does not correspond to the **economic process** that you are modeling. While it is true that economists do use this model (e.g. in labor markets) it does **not** correspond with the standard economic theory that you were taught in 206 and that we are attempting to convert into mathematical form in 208. The reason is that Q is the **independent** variable in this model and P is the **dependent** variable. But if someone were to ask you what the quantity demanded of Washington apples depended on you wouldn't say Q, you would say P. In fact in your model the correct question to ask is, what does the **demand price** of apples (the *highest* price that you would be willing to pay for a **given quantity of apples**) depend on, to which the answer **is** Q!

What Marshall seems to have done is to **flip the axes over** so that the **horizontal intercept** of the demand curve (f(x) = 0) becomes 100 and the **horizontal intercept** of the supply curve (g(x) =0) becomes –20. (And the slopes of the demand and supply curves are -½ and ¼ respectively.) If we are going to adopt this approach then we need to remember that increases in demand and supply are associated with **rightward** shifts in the two curves. While we could adopt this approach it turns out to be very cumbersome.

So **the bottom line** is that you will just have to get used to drawing the graphs correctly whenever you set up a **mathematical** model of the economists' supply and demand system. This means that whenever you illustrate a supply and demand model that has **numerical** or **algebraic** coefficients then you **must** use the mathematician's convention and put P on the horizontal axis and Q on the vertical axis. **Only** when you are illustrating a **verbal** supply and demand exercise can you use Marshall's diagram with Q on the horizontal axis and P on the vertical axis.

Figure 1

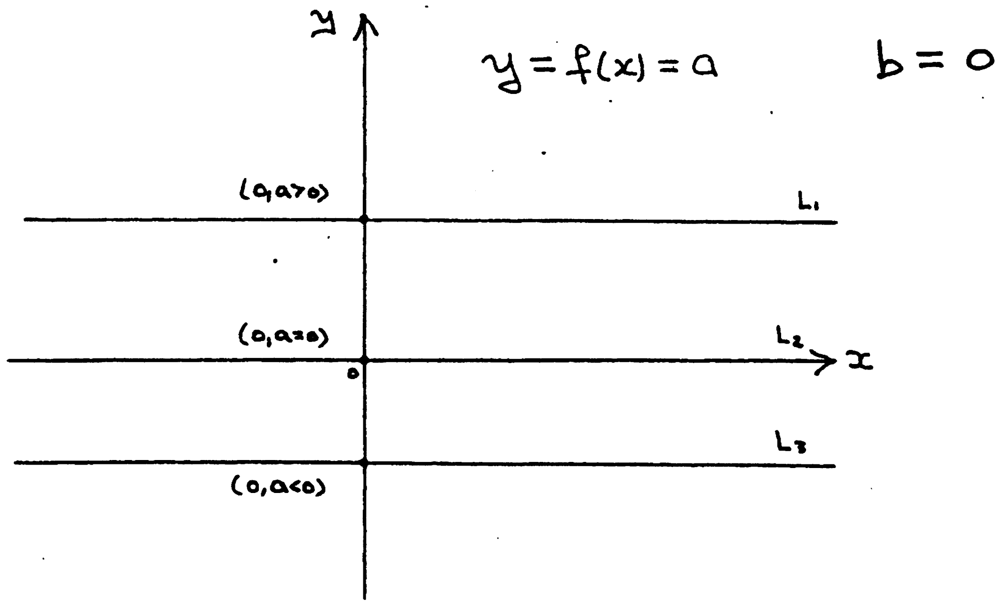

Figure 2

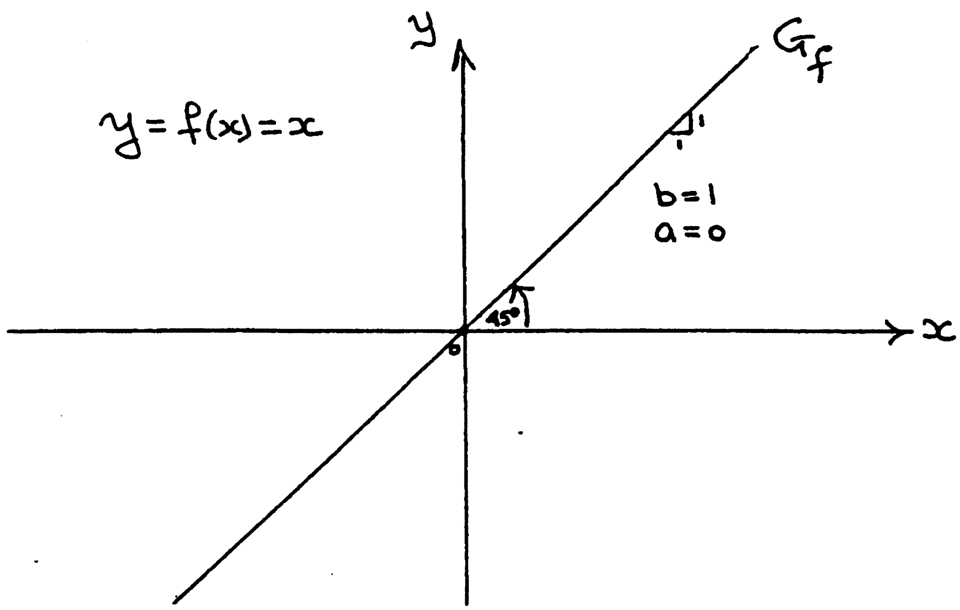

Figure 3

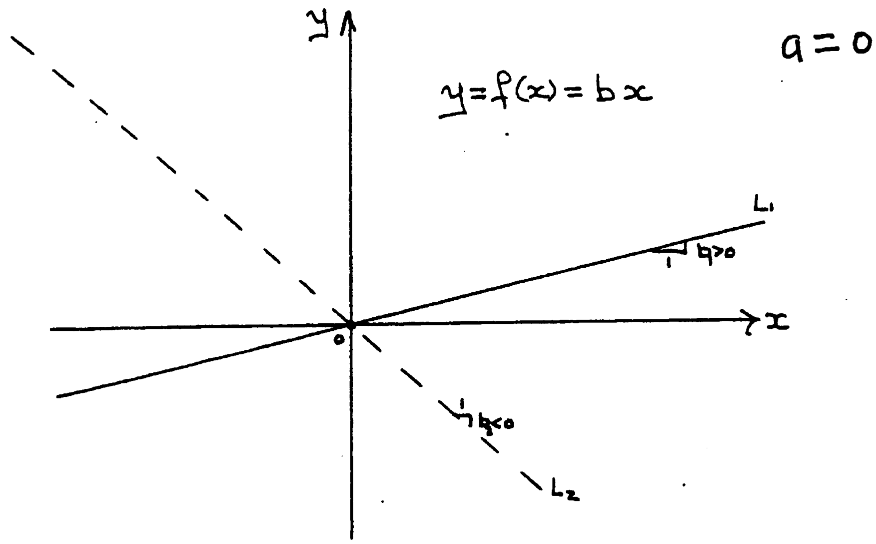

Figure 4

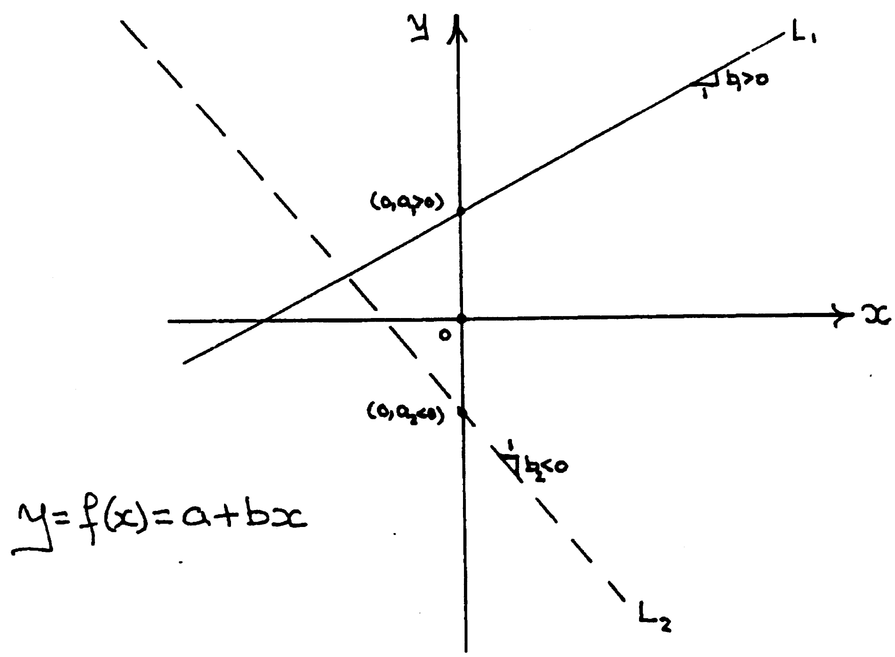

Figure 5

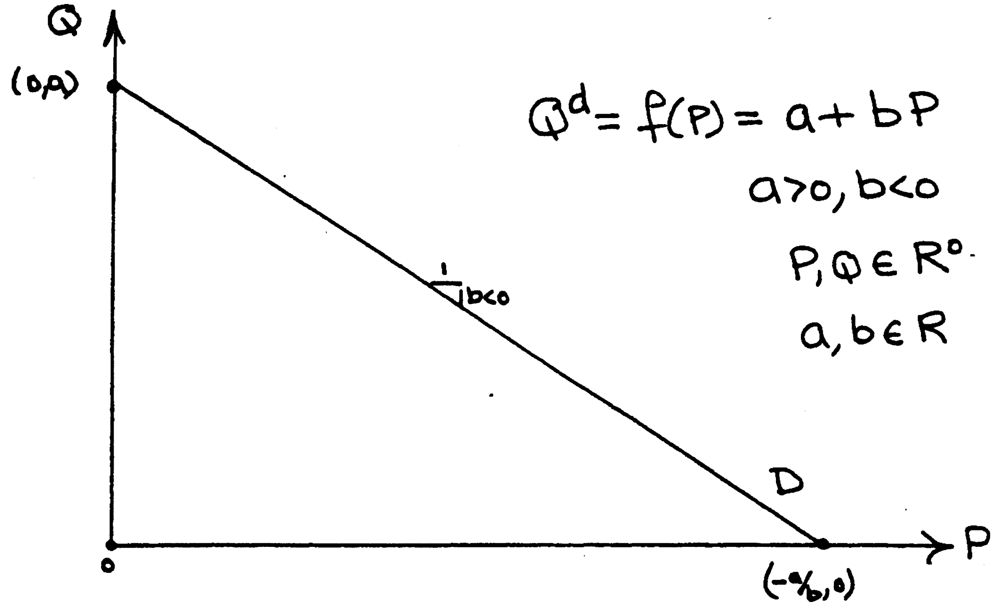

Figure 6