MATHEMATICAL REVIEW 1

SETS AND NUMBERS

1. VARIABLES

A **variable** is something that can take on more than one value, as opposed to a constant or parameter that has a single value. Variables "live" in **sets**, called their **domains**. The variable can take **any** value in the set, i.e. it can be replaced by any of the elements in the domain set.

We represent variables by *symbols*, upper or lower case letters. So we can talk about the variable P that is "really" only the *symbol that stands in for,* or*, represents* the price variable in our model. Notice how abstract this is! You should also be aware that there may be vast differences between the **theoretical** variables we use in our models, such as the quantity demanded by the consumer at a particular price (something which is intrinsically unobservable) and the corresponding **empirical** variables that we might use in an econometric model, i.e. the observed amount transacted (see A3).

$P \in R^0$ means that the variable P can take any value that is an element of the set of non-negative real numbers, i.e.

$$R^0 = \{x: x \in R \text{ and } x \geq 0\}.$$

So we can say P = 2 (because 2 > 0 and therefore $2 \in R^0$) but not P = -2 (because -2 < 0 and therefore $-2 \notin R^0$). (We can also set P = 0 because $0 \in R^0$ but that is not an interesting case to an economist.) Note that we are implicitly assuming that P is measured in, say, dollars and so 2 really stands for **$2 per unit** of the commodity. But to understand this paragraph you first need to know something about sets!

2. **SETS**

Our interest in sets is largely confined to terminology and notation. Although set theory is used extensively in modern mathematical economics we will not need any set theory in this course.

A set is strictly undefined. We will think of sets as mathematical objects which are made up of collections or groups of things possessing *some common-property,* e.g., B, the set of all commodity bundles which a consumer could purchase given a finite money income, Y, and positive, finite, prices; or, R, the set of non-negative real numbers.

Membership is the key idea when thinking about sets. We will usually label ***sets*** using capital letters (R, A, B etc.) and elements, or members, of a set by lower case letters (a, b, c, etc.). The notation $\in$ is used to indicate **membership** of a set, while its negation, $\notin$, is used to denote non-membership of a set.

$a \in A$ is read as "a is a member, or element, of the set A" or "a belongs to the set A." $b \notin A$ means that b does not belong to the set A; b is **not** an element of A.

We can visualize sets in terms of: "cloud diagrams", "line segments", or "points in the plane".

Figure 1 goes about here.

3. **NUMBERS**

There are some sets of numbers that you should be familiar with.

The set of **NATURAL** or counting numbers is the set of *positive whole* numbers **N** = {1, 2, 3, ...}. The only thing you can do with natural numbers is to add them.

If we want to do subtraction then we need a larger number set, the set of **INTEGERS**: **I** = {0, $\pm$1, $\pm$2, ...}. For example,

$$x + 3 = 2 \Rightarrow x = -1 \quad \text{but } x = -1 \notin N.$$

Division requires a further extension of the number system to take account of fractions (ratios/quotients). The set of **RATIONAL** numbers, Q, is the set of all numbers which can be written as the **ratio** of two integers; i.e.

$$Q = \{x: x = p/q, \text{ where } p \in I \text{ and } q \in I \text{ and } q \neq 0\}.$$

Taking square roots leads to numbers such as $\sqrt{2}$, which are not expressible as the ratio of two integers. The set of **REAL** numbers, **R**, the set of *ration*al and ir*ratio*nal numbers, is the set of **all** numbers that can be written as infinite decimal expansions; e.g., 2 = 2.000... = 1.999... and e = 2.7182818284... . The set of real numbers is **dense**; it has no holes or gaps. The set of real numbers can be put into one to one correspondence with the points on the real line; i.e., every real number corresponds to a point on the line, and every point on the real line corresponds to a real number.

In economics the variables we work with usually take *nonnegative* (positive or zero), or *strictly positive,* values. The set of **non-**

negative real numbers will be symbolized by R^0, the set of **positive real numbers** by R^+.

4. SOME PROPERTIES of the SET of REAL NUMBERS.

Many ECON 208 students have problems with the algebraic manipulations in the latter part of the course because they have not mastered the following elementary rules. Be certain that *you* have completely mastered these rules.

My computer skills are quite primitive, e.g. I do not know how to draw graphs that I can insert into the Manual electronically and I am using Word and occasionally the Equation Editor to put this document together – which can be very tedious. **So a.b = ab means "a times b"** and **"a/b means a divided by b".** (And use dy/dx to write derivatives). I use a lot of parentheses and bold etc. to try and make things easier for you to follow. Give me feedback about possible improvements in the format of the Manual, and hunt for typos (and even "thinkos", given my advanced age).

Division by zero is undefined, i.e. dividing by zero is mathematically meaningless. In particular, if $a \in R$ then a/0 does not mean "infinity" because the symbol ∞ is not a number but a short hand way of making the statement that some process can be extended without bound, i.e. indefinitely.

Some Basic Rules

1.) If $a,b \in R$ then $a + b = b + a$ and $ab = ba$ therefore $P.Q = Q.P$ and $p + y = y + p$. This rule is the so-called *commutative* law of real numbers. The commutative property of real numbers is very convenient because it allows you to rearrange algebraic expressions without changing their meaning.

Note that, in general, $a-b \neq b-a$ and that $a/b \neq b/a$. Further, while $(x + y)/z = x/z + y/z$, in general, $z/(x + y) \neq z/x + z/y$.
For example: $3-2 = 1 \neq 2-3 = -1$ and $2/3 \neq 3/2$ and $(2 + 4)/3 = 6/3 = 2 = 2/3 + 4/3$, and $4/(5 + 1) = 4/6 \neq 4/5 + 4/1 = 4/5 + 4$.

2.) If $a \in R$ and $a \neq 0$ then $a/a = 1$, and if $a \in R$ then $a.1 = a$. Verbally: dividing any number (except zero) by itself always yields one, and multiplying any number by one does not change the number. (These are the so called *identity* laws for real numbers.)

For example *if* $P/P = 1$ *and* $(dP/dQ).Q$ *then* $[(dP/dQ).Q].1 = [(dP/dQ).Q].(P/P) = (dP/dQ)[(Q.P)/P] = (dP/dQ)(Q/P).P$ – check that you understand this! [To save space I often write derivatives as dy/dx rather than dy over dx, and I often write several equalities on a single line. **You should re-write the math in conventional form especially if your math is rusty.]** Notice that the previous derivation simply **rearranged** terms using that useful commutative property of real numbers.

[dP/dQ represents the derivative of P with respect to Q, something that we will learn more about when we get to the second part of the course that is devoted to applications of simple differential calculus to elementary economics. If you have never done any calculus, or if your calculus is rusty, or was never any good, then ***this*** is the point at which you should start looking through the Mathematical Reviews at the end of the Manual and the last half of the Assignments and the recommended chapters in the A&L Manual. **Don't put this off** until we actually get to calculus in the lectures because those lectures **assume** that you are familiar with the basic concepts of differential calculus and provide only a very brief refresher for the material we will cover. Note that we will not do any integral calculus, not because integral calculus is not used in economics but because we do not have time to cover that material in our ten weeks.]

3.) You should also be able to **factor** (very) simple algebraic expressions by using the so called *distributive* rules:

$a(b + c) = ab + ac$ and $(a + b)c = ac + bc$

For example:

$2(3 + 4) = 2.3 + 2.4 = 6 + 8 = 14$ and $(2 + 3)4 = 2.4 + 3.4 = 8 + 12 = 20$

$Y - c(1-t)Y - m(1-t)Y = Y[1 - c(1-t) - m(1-t)]$

$(dQ/dP)(P/Q)P + P = [(dQ/dP)(P/Q)]P + (1)P =$
$[(dQ/dP)(P/Q) + 1]P = P[1 + (dQ/dP)(P/Q)]$.

You may recognize $(dQ/dP)(P/Q)$ as the calculus formula for *price elasticity of demand*: PED. Since we will spend some days later in the course talking about elasticity you should refresh your knowledge of that concept by looking at your micro Principles text and your notes from Econ 206 either now or later.

4.) If $a,b,c \in R$, and $a = bc$ and if $b \neq 0$ is a constant, then **a** is a constant *if and only* if **c** is a constant.

For example if PED = b(P/Q) where b is a constant (and PED stands for the price elasticity of demand) then PED can only be a constant if P/Q is a constant – check that you understand the algebraic argument even though you may not follow the economic reasoning.

5.) $(a/b).(c/d) = ac/bd$ therefore $(Q).(P/Q) = (Q.P)/Q = (P.Q)/Q = P$ where $Q \neq 0$ (we usually write $P.Q = Q.P$ as PQ).

6.) If $a,b \in R$ then $a.b > 0$ if and only if a and b have the same sign. $a.b < 0$ if a and b are of opposite signs. $a.b = 0$ if one or both of a and b are zero.

The Rules of Exponents (what to do with **variables** raised to **powers**).

1.) If $a \in R$ then $a^1 = a$, $a^2 = a.a$, and $a^n = a.a. \ldots .a$ (n times).

2.) $a^0 = 1$ ($a \neq 0$) by *definition.*

3.) If $a \in R$ and $a \neq 0$ then $a^{-1} = 1/a$, $a^{-2} = 1/a^2$, i.e. a negative power denotes a *reciprocal*, e.g. $2^{-1} = ½$ and $2^{-2} = ½^2 = 1/4$.

4.) $a^{1/2} = \sqrt{a}$ and $a^{1/n} = \sqrt[n]{a}$ ($a > 0$ if n is even). That is, raising a variable to a fractional power means "calculate the corresponding root" so that $a^{1/2}$ is the (positive) **square** root of a, $a^{1/3}$ is the **cube** root of a, etc. and $a^{1/n}$ is the "**nth**" root of a.

5.) If $a \in R$ then $a^{m/n} = \sqrt[n]{a^m}$ ($a^m > 0$ if n is even).

Therefore $a^{m/n}$ is the "n^{th}" root of a raised to the power m, e.g. $a^{2/3} = \sqrt[3]{a^2}$ and $a^{-4/5}$ = the reciprocal of $\sqrt[5]{a^4} = 1/(\sqrt[5]{a^4})$.

6.) If $a \in R$ then $a^m.a^n = a^{m+n}$.

Therefore $a^2.a^3 = a^{2+3} = a^5 = (a.a)(a.a.a)$

and $L^{a-1} = L^a L^{-1} = L^a/L$.

7.) $a^m/a^n = a^{m-n}$ $(a\neq 0)$.

Therefore $a^3/a^2 = a^{3-2} = a^1 = a$

and $L^{a-b} = L^a/L^b$.

8.) $(a^m)^n = a^{mn}$.
Therefore $(a^2)^3 = (a^3)^2 = (a.a.a)^2 = (a.a.a)(a.a.a) = a.a.a.a.a.a = a^6$

and $(L^a)^b = L^{ab}$.

Further if we define Q^* to be equal to $aAL^{a-1}K^b$ then (using the commutative property of real numbers) $Q^* = aAL^aL^{-1}K^b = aAL^aK^bL^{-1} = (aAL^aK^b)L^{-1} = (aAL^aK^b)(1/L)$ (using the fact that $L^{-1} = 1/L$), and therefore $Q^* = a(AL^aK^b/L)$. But $AL^aK^b = Q$ if we are dealing with the so-called Cobb-Douglas production function, and so $Q^* = Q/L$. Later we will be able to use calculus to show that Q^* is the marginal product of labor, MP_L, which is equal to $\partial Q/\partial L$. And since Q/L is the average product of labor then we have just shown, by straight algebraic manipulation, that for a Cobb-Douglas form of the production function the $MP_L = aAP_L$.

You must be able to follow simple chains of algebraic reasoning like this if you are going to be able to do well in Econ 208. **Presentation of the algebra in paragraph form is very common in economics and is used extensively in the Manual, the assignment keys and the lectures**. As I have written in the section on "How to Read the Manual. ..." you may find it easier to follow the mathematical argument if you take out the words to expose the algebraic skeleton:

$Q^* = aAL^{a-1}K^b$

change of that variable, – both represented by real numbers – is positive, negative or zero. $a > 0$ can be read "a is positive" $\Leftrightarrow$ "a is greater than zero" (where the symbol $\Leftrightarrow$ stands for "is equivalent to" or "is identical with"). Manipulation of inequalities is important in qualitative comparative static analysis – the sort of economic theory that you have been doing and will continue to do throughout most of your undergraduate work in economics. In Qualitative Comparative Statics (QCS) we are most interested in **signing** changes in variables, e.g. $\Delta P^e > 0$ or $\Delta Q^e < 0$ or $\Delta Y^e = 0$.

2.) $a > b$ means that a is *greater than* or *larger than* b. $a > b$ means that a is *to the right of* b on the *real number line*. (Vice versa for $a < b$.)

3.) Remember that **negative numbers** can be thought of as the "numerical" part of the number multiplied by minus one.

For example: $-5 = (-1)5$.

And if $a - b$ with $b < 0$ then $a - b = a + (-1)b = a + (-b)$, e.g. $5 - (-2) = 5 + (-1)(-2) = 5 + 2 = 7$. (*Subtracting* a **negative** quantity is equivalent to *adding* the "numerical" part of the number.)

4.) **Negative numbers can be tricky**, e.g. $-1/2 > -10^6$ in fact minus one half is two million times larger than minus one million. Therefore if the slope of demand curve A is -2 and the slope of demand curve B is -10 then A has a greater slope than B even though B is steeper than A, because $-2 > -10$. (Slope is an algebraic measure whereas steepness is an intuitive appraisal of geometric descent.)

5.) If $a \in R$ and if $a > 0$ and b and 1/b are also positive then $ab > 0$ and $a/b > 0$; but if $a \in R$ and if $a > 0$ but b and 1/b are negative

$= aAL^aL^{-1}K^b$ (using the commutative property of real numbers)
$= aAL^aK^bL^{-1}$ (using the commutative property of real numbers)
$= (aAL^aK^b)L^{-1}$ (using the commutative property of real numbers)
$= (aAL^aK^b)(1/L)$ (using the fact that $L^{-1} = 1/L$)
$Q^* = a(AL^aK^b/L)$ (using the commutative property of real numbers)
But $Q = AL^aK^b$ and so substituting we obtain:
$Q^* = a(AL^aK^b/L) = a(Q/L)$.
Once you have exposed the structure of the algebra you can easily return it to paragraph form if you need to.

The Signs of Ratios

If $d,n \in R$ and $d,n \neq 0$ and $r = n/d$ then $r > 0$ if n and d have the *same* sign, and $r < 0$ if n and d are of *opposite* sign. Further if $d,n \in R$ and $d,n > 0$ (i.e. $d,n \in R^+$) then $r > 1$ if $n > d$, $r = 1$ if $n = d$, and $r < 1$ if $n < d$. This says that the **ratio** of two real numbers (the *denominator* being non-zero) will be **positive** if they have the **same sign** (either both positive or both negative) and the ratio will be **negative** if the *denominator* and *numerator* are of **opposite sign**, (one positive and one negative).

Inequalities and Absolute Values

1.) The mathematical term ***sign*** is very important in qualitative economic analysis because all we can say about qualitative changes is that they are **positive** (> 0), *negative* (< 0), or zero (0) – which means that something is **increasing**, *decreasing*, or unchanged. So we may write $\Delta P > 0$ to indicate that the price has increased or $\Delta a < 0$ to indicate that the constant a has been decreased, or $\Delta Y = 0$ if Y does not change.

When you are asked to ***sign*** an effect in ECON 208 then you are being asked whether the variable in question or the rate of

(< 0) then $ab < 0$ and $a/b < 0$, i.e. **multiplying/dividing an inequality by a negative number changes the *sign* of the inequality**.

For example, $2 > 0$ and $2.3 = 6 > 0$ but $2.(-3) = -6 < 0$ and $-2 < 0$ and $2/(-3) = -2/3 < 0$.

If $k > 0$ (where k is some macroeconomic "multiplier") then $-k < 0$.

If $-1 > 1/(PED)$ where $PED < 0$ then $-1.PED < 1$ (we changed sign because PED is negative) or $PED > 1/-1$ or $PED > -1$ or $0 > PED > -1$.

6.) Finally you need to remember that the "**absolute value**" of a variable is defined by $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a < 0$. That is the **absolute** value of any real number is its "**numerical"** value.

For example $|2| = 2$ and $|-2| = -(-2) = 2$.

And if $|k'| = k - 1$ where $k' < 0$ then $k' = -(k-1) = -k + 1$.

Final Thoughts

How much of this do you need to know **today**? Not much. But as we progress through the quarter I will be doing manipulations that are based on these simple mathematical facts, and while I will remind you of the basic definitions and rules as we encounter them, if you find this sort of thing overwhelming – if your poor brain turns to mush when you think about this sort of thing – then you need to brush up your algebra skills and, if necessary, put off ECON 208 until you have sat in on MATH 156 or MATH 157. This really is **very** basic stuff – you all once knew how to do this in high school (I hope!) – and it looks more daunting than it really is because I have written down a lot of rules and definitions that I assume you were once familiar with without spending pages (and a great deal of my valuable time) giving you numerous examples. The algebra section of the math section of the library has plenty of books that you can borrow that will provide you with the necessary examples and more elaborate explanations if you need them and A&L review some of this material in Ch. 2.12- 13. There are also many popular books and other aids designed to refresh those rusty algebra skills that you can buy second hand or cheaply from the bookstore.

What you need to do is to be certain that by the time we come to use some of this stuff that you will be on top of it, or *at least know where in the Manual to look for a statement of the rules and at least one example of their use*.

Incidentally, there are two other parts of basic algebra that you will encounter fairly frequently in your upper division courses – logarithms and exponentials. While I would like to review this material – and used to ten years ago! – I have decided not to because we will not use these tools in ECON 208 but they are covered in A&L 2.14, 13.6-7, and 13.11.

Figure 1